Chapter 9 Matrices and Determinants

9.1 Matrices and Systems of Equations

9.1 Practice Problems

1. a. 3×2 **b.** 1×2

2. $\left[\begin{array}{ccc|c} 0 & 3 & -1 & 8 \\ 1 & 4 & 0 & 14 \\ 0 & -2 & 9 & 0 \end{array}\right]$

3. $\begin{bmatrix} 3 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 4 & 6 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1}$
$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow{-3R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \end{bmatrix}$

4. First write the system as an augmented matrix.

$$\begin{cases} x - 6y + 3z = -2 \\ 3x + 3y - 2z = -2 \\ 2x - 3y + z = -2 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & -6 & 3 & -2 \\ 3 & 3 & -2 & -2 \\ 2 & -3 & 1 & -2 \end{array}\right]$$

Now perform the row operations.

$$\left[\begin{array}{ccc|c} 1 & -6 & 3 & -2 \\ 3 & 3 & -2 & -2 \\ 2 & -3 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[2R_1 - R_3 \to R_3]{3R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -6 & 3 & -2 \\ 0 & -21 & 11 & -4 \\ 0 & -9 & 5 & -2 \end{array}\right]$$

$$\xrightarrow{9R_2 - 21R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -6 & 3 & -2 \\ 0 & -21 & 11 & -4 \\ 0 & 0 & -6 & 6 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{6}R_3 \to R_3]{-\frac{1}{21}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -6 & 3 & -2 \\ 0 & 1 & -\frac{11}{21} & \frac{4}{21} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Thus, $z = -1$.
Using back-substitution, we have
$y - \frac{11}{21}(-1) = \frac{4}{21} \Rightarrow y = -\frac{7}{21} = -\frac{1}{3}$. Then,
$x - 6\left(-\frac{1}{3}\right) + 3(-1) = -2 \Rightarrow x - 1 = -2 \Rightarrow$
$x = -1$

The solution is $\left\{\left(-1, -\frac{1}{3}, -1\right)\right\}$.

5. First write the system as an augmented matrix.

$$\begin{cases} x - 2y = 1 \\ 2x + 3y = 16 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 3 & 16 \end{array}\right]$$

Now perform the row operations.

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 2 & 3 & 16 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 7 & 14 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Thus, $y = 2$. Using back-substitution, we have $x - 2(2) = 1 \Rightarrow x = 5$. The solution is $\{(5, 2)\}$.

6. First write the system as an augmented matrix.

$$\begin{cases} 2x + y - z = 7 \\ x - 3y - 3z = 4 \\ 4x + y + z = 3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 1 & -3 & -3 & 4 \\ 4 & 1 & 1 & 3 \end{array}\right]$$

Now perform the row operations.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 7 \\ 1 & -3 & -3 & 4 \\ 4 & 1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -3 & -3 & 4 \\ 2 & 1 & -1 & 7 \\ 4 & 1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[-4R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & -3 & 4 \\ 0 & 7 & 5 & -1 \\ 0 & 13 & 13 & -13 \end{array}\right]$$

$$\xrightarrow[\frac{1}{13}R_3 \to R_3]{\frac{1}{7}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -3 & -3 & 4 \\ 0 & 1 & \frac{5}{7} & -\frac{1}{7} \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{\frac{7}{2}(-R_2 + R_3) \to R_3} \left[\begin{array}{ccc|c} 1 & -3 & -3 & 4 \\ 0 & 1 & \frac{5}{7} & -\frac{1}{7} \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Thus, $z = -3$. Using back-substitution, we have $y + \frac{5}{7}(-3) = -\frac{1}{7} \Rightarrow y = 2$ and
$x - 3(2) - 3(-3) = 4 \Rightarrow x = 1$.
The solution is $\{(1, 2, -3)\}$.

7. $\begin{cases} 6x+8y-14z=3 \\ 3x+4y-\ \ 7z=12 \\ 6x+3y+\ \ \ z=0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 6 & 8 & -14 & 3 \\ 3 & 4 & -7 & 12 \\ 6 & 3 & 1 & 0 \end{array}\right]$

Now perform the row operations.

$$\left[\begin{array}{ccc|c} 6 & 8 & -14 & 3 \\ 3 & 4 & -7 & 12 \\ 6 & 3 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & 4 & -7 & 12 \\ 6 & 8 & -14 & 3 \\ 6 & 3 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & \frac{4}{3} & -\frac{7}{3} & 4 \\ 6 & 8 & -14 & 3 \\ 6 & 3 & 1 & 0 \end{array}\right] \xrightarrow{-6R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & \frac{4}{3} & -\frac{7}{3} & 4 \\ 0 & 0 & 0 & -21 \\ 6 & 3 & 1 & 0 \end{array}\right]$$

Since the second row is equivalent to $0 = -21$, the system is inconsistent and the solution is $\varnothing$.

8. $\begin{cases} 2x-3y-2z=0 \\ \ \ x+\ \ y-2z=7 \\ 3x-5y-5z=3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2 & -3 & -2 & 0 \\ 1 & 1 & -2 & 7 \\ 3 & -5 & -5 & 3 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 2 & -3 & -2 & 0 \\ 1 & 1 & -2 & 7 \\ 3 & -5 & -5 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 7 \\ 2 & -3 & -2 & 0 \\ 3 & -5 & -5 & 3 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 7 \\ 0 & -5 & 2 & -14 \\ 0 & -8 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 7 \\ 0 & 1 & -\frac{2}{5} & \frac{14}{5} \\ 0 & -8 & 1 & -18 \end{array}\right] \xrightarrow[8R_2+R_3 \to R_3]{R_1-R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{8}{5} & \frac{21}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{14}{5} \\ 0 & 0 & -\frac{11}{5} & \frac{22}{5} \end{array}\right]$$

$$\xrightarrow{-\frac{5}{11}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{8}{5} & \frac{21}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{14}{5} \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_2+\frac{2}{5}R_3 \to R_2]{R_1+\frac{8}{5}R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The solution is $\{(1, 2, -2)\}$.

9. $\begin{cases} x+\qquad\ z=-1 \\ \quad 3y+2z=\ \ 5 \\ 3x-3y+\ z=-8 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 3 & -3 & 1 & -8 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 3 & -3 & 1 & -8 \end{array}\right] \xrightarrow{-3R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 3 & 2 & 5 \\ 0 & -3 & -2 & -5 \end{array}\right] \xrightarrow[R_2+R_3 \to R_3]{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The matrix is in row echelon form. The equivalent system is $\begin{cases} x+z=-1 \\ y+\frac{2}{3}z=\frac{5}{3} \end{cases}$. Solving for x and y in terms of z, we have $x=-z-1$ and $y=\frac{5}{3}-\frac{2}{3}z$.

The solution set is $\left\{\left(-z-1, \frac{5}{3}-\frac{2}{3}z, z\right)\right\}$.

10. a. $\begin{cases} s=0.01s+\ \ 0.1c+0.1t+12.46 \\ c=\ \ 0.2s+0.02c+0.01t+3 \\ t=0.25s+\ \ 0.3c+\qquad\quad 2.7 \end{cases}$ or

$\begin{cases} 0.99s\ \ -0.1c\ \ -0.1t=12.46 \\ -0.2s+0.98c-0.01t=3 \\ -0.25s\ \ -0.3c\qquad +t=2.7 \end{cases}$

b. $0.99(14)-0.1(6)-0.1(8)=12.46$ ✓
$-0.2(14)+0.98(6)-0.01(8)=3$ ✓
$-0.25(14)-0.3(6)+(8)=2.7$ ✓

9.1 Basic Concepts and Skills

1. A matrix is any rectangular array of numbers.
2. The array of coefficients and constants in a linear system is called the augmented matrix of the system.
3. Two matrices are row equivalent if one can be obtained form the other by a sequence of row operations.
4. If a matrix is in row-echelon form and each leading entry 1 is the only nonzero entry in its column, then the matrix is in reduced row echelon form.
5. False. A 3×4 matrix has three rows with four entries in each row. In other words, the matrix has three rows and four columns.
6. True
7. False. The augmented matrix of the system is $\left[\begin{array}{ccc|c} 2 & 3 & 0 & 5 \\ 0 & 3 & -4 & -1 \\ 1 & 0 & 2 & 3 \end{array}\right]$.
8. False. The augmented matrix of the system is $\left[\begin{array}{cc|c|c} 2 & -1 & 1 \\ 3 & 1 & 9 \\ 5 & 02 & 4 \end{array}\right]$.

9. 1×1 **10.** 1×4 **11.** 2×4

12. 3×3 **13.** 2×3 **14.** 3×4

15. $a_{13} = 3, a_{31} = 9, a_{33} = 11, a_{34} = 12$

16. a. a_{23} **b.** a_{32}

c. a_{14} **d.** a_{34}

17. No, the array is not a matrix because it is not a rectangular array of numbers.

18. $\begin{bmatrix} 6 & -5 & 5 \\ 7 & 2 & 4 \end{bmatrix}$ **19.** $\left[\begin{array}{cc|c} 2 & 4 & 2 \\ 1 & -3 & 1 \end{array}\right]$

20. $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 3 & 5 & 11 \end{array}\right]$ **21.** $\left[\begin{array}{cc|c} 5 & -7 & 11 \\ -13 & 17 & 19 \end{array}\right]$

22. $\left[\begin{array}{cc|c} -3 & 2 & 10 \\ 5 & -1 & -7 \end{array}\right]$ **23.** $\left[\begin{array}{ccc|c} -1 & 2 & 3 & 8 \\ 2 & -3 & 9 & 16 \\ 4 & -5 & -6 & 32 \end{array}\right]$

24. $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & 0 & 3 & -5 \\ 0 & 1 & -2 & 7 \end{array}\right]$ **25.** $\begin{cases} x + 2y - 3z = 4 \\ -2x - 3y + z = 5 \\ 3x - 3y + 2z = 7 \end{cases}$

26. $\begin{cases} -x + 2y + 3z = 6 \\ 2x + 3y + z = 2 \\ 4x + 3y + 2z = 1 \end{cases}$ **27.** $\begin{cases} x - y + z = 2 \\ 2x + y - 3z = 6 \end{cases}$

28. $\begin{cases} x + y + z = 2 \\ x - y - z = 4 \\ 2x + 3y + z = 6 \\ -x + y - z = 8 \end{cases}$

29.(i) $\begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$

30.(i) $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 7 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 7 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 5 & 7 \end{bmatrix} \xrightarrow{-R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 6 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

31.(i) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 4 & -3 & 11 \\ 0 & 1 & 5 & -3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & -3 \\ 0 & 4 & -3 & 11 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & -3 \\ 0 & 4 & -3 & 11 \end{bmatrix} \xrightarrow{-4R_2 + R_3 \to R_3}$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & -23 & 23 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & -23 & 23 \end{bmatrix} \xrightarrow{-\frac{1}{23}R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

32.(i) $\begin{bmatrix} 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 2 & 3 & 4 \\ -2 & -1 & 7 & 3 \end{bmatrix} \xrightarrow{2R_1+R_3 \to R_3} \begin{bmatrix} 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2+R_3 \to R_3} \begin{bmatrix} 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 2 & 3 & 4 \\ 0 & 3 & \frac{9}{2} & 6 \end{bmatrix}$

33. $\begin{bmatrix} 4 & 5 & -7 \\ 5 & 4 & -2 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & \frac{5}{4} & -\frac{7}{4} \\ 5 & 4 & -2 \end{bmatrix}$

$\xrightarrow{-5R_1+R_2 \to R_2} \begin{bmatrix} 1 & \frac{5}{4} & -\frac{7}{4} \\ 0 & -\frac{9}{4} & \frac{27}{4} \end{bmatrix}$

$\xrightarrow{-\frac{4}{9}R_2 \to R_2} \begin{bmatrix} 1 & \frac{5}{4} & -\frac{7}{4} \\ 0 & 1 & -3 \end{bmatrix}$

34. $\begin{bmatrix} 2 & 6 & -8 \\ 3 & -1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 3 & -4 \\ 3 & -1 & 2 \end{bmatrix}$

$\xrightarrow{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & -4 \\ 0 & -10 & 14 \end{bmatrix}$

$\xrightarrow{-\frac{1}{10}R_2} \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -\frac{7}{5} \end{bmatrix}$

35. $\begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & -3 & -2 & 0 \\ 0 & 7 & 5 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 7 & 5 & -3 \end{bmatrix}$

$\xrightarrow{-7R_2+R_3 \to R_3} \begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & -3 \end{bmatrix}$

$\xrightarrow{3R_3} \begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & -9 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 8 \\ 1 & -3 & -2 & 5 \end{bmatrix}$

$\xrightarrow{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 1 & -3 & -2 & 5 \end{bmatrix}$

$\xrightarrow{-R_1+R_3 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & -4 & -3 & 2 \end{bmatrix}$

$\xrightarrow{4R_2+R_3 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 10 \end{bmatrix}$

37. No. The matrix does not have a step-like pattern that moves down and to the right. (Property 2)

38. No. The leading term of each nonzero row is not 1. (Property 1)

39. Yes. The matrix is in reduced row-echelon form.

40. No. The matrix does not have a step-like pattern that moves down and to the right. (Property 2)

41. Yes. The matrix is in reduced row-echelon form.

42. Yes. The matrix is in reduced row-echelon form.

43. Yes. The matrix is in reduced row-echelon form.

44. No. The matrix does not have a step-like pattern that moves down and to the right. (Property 2), and the leading term of each non-zero row (row 3) is not 1 (Property 1).

45. $\begin{cases} x+2y=1 \\ \quad\ \ y=-2 \end{cases} \Rightarrow x+2(-2)=1 \Rightarrow x=5$

The solution is $\{(5, -2)\}$.

46. $\begin{cases} x=2 \\ y=3 \end{cases}$

The solution is $\{(2, 3)\}$.

47. $\begin{cases} x+4y+2z=2 \\ \qquad\quad\ \ z=3 \end{cases} \Rightarrow x+4y=-4 \Rightarrow x=-4y-4$

The solution is $\{(-4y-4, y, 3)\}$.

48. $\begin{cases} x+2y+3z=4 \\ \qquad\quad 0z=1 \end{cases}$

The system is inconsistent.

49. $\begin{cases} x+2y+3z=2 \\ \qquad y-2z=4 \\ \qquad\quad\ \ z=-1 \end{cases} \Rightarrow y-2(-1)=4 \Rightarrow y=2$

$x+2(2)+3(-1)=2 \Rightarrow x=1$

The solution is $\{(1, 2, -1)\}$.

50. $\begin{cases} x+2z=12 \\ y+3z=12 \\ \quad\ \ z=5 \end{cases} \Rightarrow y+3(5)=12 \Rightarrow y=-3$

$x+2(5)=12 \Rightarrow x=2$

The solution is $\{(2, -3, 5)\}$.

51. $\begin{cases} x=2 \\ y=-5 \\ z+2w=3 \end{cases} \Rightarrow z=-2w+3$

The solution is $\{(2, -5, -2w+3, w)\}$.

52. $\begin{cases} x=4 \\ y=3 \\ w=2 \end{cases}$

The solution is $\{(4, 3, z, 2)\}$.

53. $\begin{cases} x \qquad\ \ =-5 \\ y \qquad\ \ =4 \\ z+2w=3 \\ \qquad w=0 \end{cases} \Rightarrow z+2(0)=3 \Rightarrow z=3$

The solution is $\{(-5, 4, 3, 0)\}$.

54. $\begin{cases} x=3 \\ y=2 \\ z=0 \\ 0w=1 \end{cases}$

The system is inconsistent.

55. $\left[\begin{array}{cc|c} 1 & -2 & 11 \\ 2 & -1 & 13 \end{array}\right] \xrightarrow{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 11 \\ 0 & 3 & -9 \end{array}\right] \xrightarrow{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 11 \\ 0 & 1 & -3 \end{array}\right] \Rightarrow$

$y=-3; x-2(-3)=11 \Rightarrow x=5$

The solution is $\{(5, -3)\}$.

56. $\left[\begin{array}{cc|c} 3 & -2 & 4 \\ 4 & -3 & 5 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{4}{3} \\ 4 & -3 & 5 \end{array}\right] \xrightarrow{-4R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] \xrightarrow{-3R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & 1 & 1 \end{array}\right] \Rightarrow$

$y=1; x-\frac{2}{3}(1)=\frac{4}{3} \Rightarrow x=2$

The solution is $\{(2, 1)\}$.

57. $\left[\begin{array}{cc|c} 2 & -3 & 3 \\ 4 & -1 & 11 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & \frac{3}{2} \\ 4 & -1 & 11 \end{array}\right] \xrightarrow{-4R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{\frac{1}{5}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 1 \end{array}\right] \Rightarrow$

$y=1; x-\frac{3}{2}(1)=\frac{3}{2} \Rightarrow x=3$

The solution is $\{(3, 1)\}$.

58. $\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 6 & 4 & 3 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 6 & 4 & 3 \end{array}\right] \xrightarrow{-6R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{array}\right] \Rightarrow$

$0=1 \Rightarrow$ there is no solution.

Solution set: $\varnothing$

59. $\left[\begin{array}{cc|c} 3 & -5 & 4 \\ 4 & -15 & 13 \end{array}\right] \xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -\frac{5}{3} & \frac{4}{3} \\ 4 & -15 & 13 \end{array}\right]$

$\xrightarrow{-4R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{5}{3} & \frac{4}{3} \\ 0 & -\frac{25}{3} & \frac{23}{3} \end{array}\right]$

$\xrightarrow{-\frac{3}{25}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{5}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{23}{25} \end{array}\right] \Rightarrow$

$y = -\frac{23}{25}; x - \left(\frac{5}{3}\right)\left(-\frac{23}{25}\right) = \frac{4}{3} \Rightarrow x = -\frac{1}{5}$

The solution is $\left\{\left(-\frac{1}{5}, -\frac{23}{25}\right)\right\}$.

60. $\left[\begin{array}{cc|c} -2 & 4 & 1 \\ 3 & -5 & -9 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & -2 & -\frac{1}{2} \\ 3 & -5 & -9 \end{array}\right]$

$\xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & -\frac{1}{2} \\ 0 & 1 & -\frac{15}{2} \end{array}\right] \Rightarrow$

$y = -\frac{15}{2}; x - 2\left(-\frac{15}{2}\right) = -\frac{1}{2} \Rightarrow x = -\frac{31}{2}$

The solution is $\left\{\left(-\frac{31}{2}, -\frac{15}{2}\right)\right\}$.

61. $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 3 & -4 & 2 \end{array}\right] \xrightarrow[-3R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{array}\right]$

$\xrightarrow[-R_3 \to R_3]{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow$

$y = 1; x - 1 = 1 \Rightarrow x = 2$

The solution is $\{(2, 1)\}$.

62. Rewrite the system as $\begin{cases} -2x + y = 1 \\ 3x + 2y = -1.5 \\ 4x - 2y = -2 \end{cases}$.

$\left[\begin{array}{cc|c} -2 & 1 & 1 \\ 3 & 2 & -1.5 \\ 4 & -2 & -2 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 3 & 2 & -1.5 \\ 4 & -2 & -2 \end{array}\right]$

$\xrightarrow[R_1 - \frac{1}{4}R_3 \to R_3]{R_1 - \frac{1}{3}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{7}{6} & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{-\frac{6}{7}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$

$y = 0; x - 0 = -\frac{1}{2} \Rightarrow x = -\frac{1}{2}$

The solution is $\left\{\left(-\frac{1}{2}, 0\right)\right\}$.

63. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{array}\right]$

$\xrightarrow[-2R_1 + R_3 \to R_3]{R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 0 & 4 \\ 0 & -1 & -3 & -11 \end{array}\right]$

$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & -3 & -11 \end{array}\right]$

$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -9 \end{array}\right]$

$\xrightarrow{-\frac{1}{3}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow$

$z = 3; y = 2; x + 2 + 3 = 6 \Rightarrow x = 1$

The solution is $\{(1, 2, 3)\}$.

64. $\left[\begin{array}{ccc|c} 2 & 4 & 1 & 5 \\ 1 & 1 & 1 & 6 \\ 2 & 3 & 1 & 6 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & 5 \\ 2 & 3 & 1 & 6 \end{array}\right]$

$\xrightarrow[-2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 2 & -1 & -7 \\ 0 & 1 & -1 & -6 \end{array}\right]$

$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & -1 & -6 \end{array}\right]$

$\xrightarrow{R_2 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{1}{2} & \frac{5}{2} \end{array}\right]$

$\xrightarrow{2R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & 5 \end{array}\right]$

$z = 5; y - \frac{1}{2}(5) = -\frac{7}{2} \Rightarrow y = -1$
$x - 1 + 5 = 6 \Rightarrow x = 2$
The solution is $\{(2, -1, 5)\}$.

65. $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 9 \\ 1 & 1 & 1 & 9 \\ 3 & -1 & -1 & -1 \end{array}\right]$

$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 3 & -1 & 9 \\ 3 & -1 & -1 & -1 \end{array}\right]$

$\xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & -3 & -9 \\ 0 & -4 & -4 & -28 \end{array}\right]$

$\xrightarrow{4R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & -3 & -9 \\ 0 & 0 & -16 & -64 \end{array}\right]$

$\xrightarrow{-\frac{1}{16}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & -3 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right] \Rightarrow$

$z = 4; y - 3(4) = -9 \Rightarrow y = 3$
$x + 3 + 4 = 9 \Rightarrow x = 2$
The solution is $\{(2, 3, 4)\}$.

66. $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{array}\right]$

$\xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -3 & -1 & 1 \\ 0 & -4 & -7 & -10 \end{array}\right]$

$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & -4 & -7 & -10 \end{array}\right]$

$\xrightarrow{4R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{17}{3} & -\frac{34}{3} \end{array}\right]$

$\xrightarrow{-\frac{3}{17}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow$

$z = 2$
$y + \left(\frac{1}{3}\right)(2) = -\frac{1}{3} \Rightarrow y = -1$
$x - 1 + 2(2) = 4 \Rightarrow x = 1$
The solution is $\{(1, -1, 2)\}$.

67. $\left[\begin{array}{ccc|c} 3 & 2 & 4 & 19 \\ 2 & -1 & 1 & 3 \\ 6 & 7 & -1 & 17 \end{array}\right]$

$\xrightarrow{R_1 \leftrightarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 3 & 2 & 4 & 19 \\ 6 & 7 & -1 & 17 \end{array}\right]$

$\xrightarrow[-6R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{7}{2} & \frac{5}{2} & \frac{29}{2} \\ 0 & 10 & -4 & 8 \end{array}\right]$

$\xrightarrow{\frac{2}{7}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{5}{7} & \frac{29}{7} \\ 0 & 10 & -4 & 8 \end{array}\right]$

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$$\xrightarrow{-10R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2}\\ 0 & 1 & \frac{5}{7} & \frac{29}{7}\\ 0 & 0 & -\frac{78}{7} & -\frac{234}{7}\end{array}\right]$$

$$\xrightarrow{-\frac{7}{78}R_3\to R_3}\left[\begin{array}{ccc|c}1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2}\\ 0 & 1 & \frac{5}{7} & \frac{29}{7}\\ 0 & 0 & 1 & 3\end{array}\right]\Rightarrow$$

$z=3$

$y+\left(\frac{5}{7}\right)(3)=\frac{29}{7}\Rightarrow y=2$

$x-\frac{1}{2}(2)+\frac{1}{2}(3)=\frac{3}{2}\Rightarrow x=1$

The solution is $\{(1, 2, 3)\}$.

68. $$\left[\begin{array}{ccc|c}4 & 3 & 1 & 8\\ 2 & 1 & 4 & -4\\ 3 & 0 & 1 & 1\end{array}\right]\xrightarrow{\frac{1}{4}R_1\to R_1}\left[\begin{array}{ccc|c}1 & \frac{3}{4} & \frac{1}{4} & 2\\ 2 & 1 & 4 & -4\\ 3 & 0 & 1 & 1\end{array}\right]$$

$$\xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\left[\begin{array}{ccc|c}1 & \frac{3}{4} & \frac{1}{4} & 2\\ 0 & -\frac{1}{2} & \frac{7}{2} & -8\\ 0 & -\frac{9}{4} & \frac{1}{4} & -5\end{array}\right]$$

$$\xrightarrow{-\frac{2}{31}\left(-\frac{9}{2}R_2+R_3\right)\to R_3}\left[\begin{array}{ccc|c}1 & \frac{3}{4} & \frac{1}{4} & 2\\ 0 & -\frac{1}{2} & \frac{7}{2} & -8\\ 0 & 0 & 1 & -2\end{array}\right]$$

$$\xrightarrow{-2R_2\to R_2}\left[\begin{array}{ccc|c}1 & \frac{3}{4} & \frac{1}{4} & 2\\ 0 & 1 & -7 & 16\\ 0 & 0 & 1 & -2\end{array}\right]\Rightarrow$$

$z=-2;\ y-7(-2)=16\Rightarrow y=2$

$x+\frac{3}{4}(2)+\frac{1}{4}(-2)=2\Rightarrow x=1$

The solution is $\{(1, 2, -2)\}$.

69. $$\left[\begin{array}{ccc|c}1 & -1 & 0 & 1\\ 1 & 0 & -1 & -1\\ 2 & 1 & -1 & 3\end{array}\right]$$

$$\xrightarrow[-2R_1+R_3\to R_3]{R_1-R_2\to R_2}\left[\begin{array}{ccc|c}1 & -1 & 0 & 1\\ 0 & -1 & 1 & 2\\ 0 & 3 & -1 & 1\end{array}\right]$$

$$\xrightarrow[3R_2+R_3\to R_3]{-R_2+R_1\to R_1}\left[\begin{array}{ccc|c}1 & 0 & -1 & -1\\ 0 & -1 & 1 & 2\\ 0 & 0 & 2 & 7\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3\to R_3}\left[\begin{array}{ccc|c}1 & 0 & -1 & -1\\ 0 & -1 & 1 & 2\\ 0 & 0 & 1 & \frac{7}{2}\end{array}\right]$$

$$\xrightarrow[-R_2+R_3\to R_2]{R_3+R_1\to R_1}\left[\begin{array}{ccc|c}1 & 0 & 0 & \frac{5}{2}\\ 0 & 1 & 0 & \frac{3}{2}\\ 0 & 0 & 1 & \frac{7}{2}\end{array}\right]\Rightarrow$$

$x=\frac{5}{2},\ y=\frac{3}{2},\ z=\frac{7}{2}$

The solution is $\left\{\left(\frac{5}{2},\frac{3}{2},\frac{7}{2}\right)\right\}$.

70. $$\left[\begin{array}{ccc|c}4 & 0 & 5 & 7\\ 0 & 1 & -6 & 8\\ 3 & 4 & 0 & 9\end{array}\right]\xrightarrow{R_1\leftrightarrow R_3}\left[\begin{array}{ccc|c}3 & 4 & 0 & 9\\ 0 & 1 & -6 & 8\\ 4 & 0 & 5 & 7\end{array}\right]$$

$$\xrightarrow[\frac{1}{5}R_3\to R_3]{\frac{1}{3}R_1\to R_1}\left[\begin{array}{ccc|c}1 & \frac{4}{3} & 0 & 3\\ 0 & 1 & -6 & 8\\ \frac{4}{5} & 0 & 1 & \frac{7}{5}\end{array}\right]$$

$$\xrightarrow{-\frac{4}{5}R_1+R_3\to R_3}\left[\begin{array}{ccc|c}1 & \frac{4}{3} & 0 & 3\\ 0 & 1 & -6 & 8\\ 0 & -\frac{16}{15} & 1 & -1\end{array}\right]$$

$$\xrightarrow{\frac{16}{15}R_2+R_3\to R_3}\left[\begin{array}{ccc|c}1 & \frac{4}{3} & 0 & 3\\ 0 & 1 & -6 & 8\\ 0 & 0 & -\frac{27}{5} & \frac{113}{15}\end{array}\right]$$

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$$\xrightarrow{-\frac{5}{27}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & \frac{4}{3} & 0 & 3 \\ 0 & 1 & -6 & 8 \\ 0 & 0 & 1 & -\frac{113}{81} \end{array}\right]$$

$$\xrightarrow{6R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & \frac{4}{3} & 0 & 3 \\ 0 & 1 & 0 & -\frac{10}{27} \\ 0 & 0 & 1 & -\frac{113}{81} \end{array}\right]$$

$$\xrightarrow{-\frac{4}{3}R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{283}{81} \\ 0 & 1 & 0 & -\frac{10}{27} \\ 0 & 0 & 1 & -\frac{113}{81} \end{array}\right] \Rightarrow$$

$x = \frac{283}{81}, y = -\frac{10}{27}, z = -\frac{113}{81}$

The solution is $\left\{\left(\frac{283}{81}, -\frac{10}{27}, -\frac{113}{81}\right)\right\}$.

71. $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 1 & 3 & 5 & 10 \\ 3 & 5 & 3 & 18 \end{array}\right]$

$$\xrightarrow[-3R_1 + R_3 \to R_3]{R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & -6 & -6 \\ 0 & 2 & 6 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & -2 & -6 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$x - 4z = 1 \Rightarrow x = 1 + 4z$

$y + 3z = 3 \Rightarrow y = 3 - 3z$

The solution is $\{(1 + 4z, 3 - 3z, z)\}$

72. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & -5 \\ 2 & -1 & -1 & -4 \\ 0 & 1 & 1 & -2 \end{array}\right]$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & -1 & -1 & -4 \\ 1 & 1 & 1 & -5 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_1} \left[\begin{array}{ccc|c} 3 & 0 & 0 & -9 \\ 1 & 1 & 1 & -5 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 1 & 1 & 1 & -5 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{-R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 1 & -2 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$y + z = -2 \Rightarrow y = -2 - z$

The solution is $\{-3, -2 - z, z\}$.

73. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 3 & 1 & 2 & 3 \\ 2 & 5 & 3 & 9 \end{array}\right]$

$$\xrightarrow[-2R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & -5 & 5 & -15 \\ 0 & 1 & 5 & -3 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 5 & -3 \end{array}\right]$$

$$\xrightarrow[R_2 - R_3 \to R_3]{-2R_2 + R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -6 & 6 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{6}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3 \to R_2]{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow$$

$x = 1, y = 2, z = -1$

The solution is $\{(1, 2, -1)\}$.

74. $\left[\begin{array}{ccc|c} 2 & 4 & -1 & 9 \\ 1 & 3 & -3 & 4 \\ 3 & 1 & 2 & 7 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & -3 & 4 \\ 2 & 4 & -1 & 9 \\ 3 & 1 & 2 & 7 \end{array}\right]$

$\xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 3 & -3 & 4 \\ 0 & -2 & 5 & 1 \\ 0 & -8 & 11 & -5 \end{array}\right]$

$\xrightarrow{-4R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 3 & -3 & 4 \\ 0 & -2 & 5 & 1 \\ 0 & 0 & -9 & -9 \end{array}\right]$

$\xrightarrow{-\frac{1}{9}R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 3 & -3 & 4 \\ 0 & -2 & 5 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\xrightarrow[-5R_3+R_2\to R_2]{3R_3+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 7 \\ 0 & -2 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\xrightarrow{-\frac{1}{2}R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\xrightarrow{-3R_2+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$

The solution is {(1, 2, 1)}.

75. a. There is one solution: {(2, 3)}.

b. There are infinitely many solutions: $\{(-2y+2, y, 2)\}$.

c. There is no solution.

76. a. There is one solution: {(−1, 2, 3)}.

b. There is no solution.

c. There are infinitely many solutions.

77. False. Each row of matrix *A* has seven entries.

78. False. For example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is in reduced row-echelon form.

9.1 Applying the Concepts

79. a. $\begin{cases} a = 0.1b + 1000 \\ b = 0.2a + 780 \end{cases}$

b. Rewrite the system as

$\begin{cases} a - 0.1b = 1000 \\ -0.2a + b = 780 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & -0.1 & 1000 \\ -0.2 & 1 & 780 \end{array}\right]$

c. $\left[\begin{array}{cc|c} 1 & -0.1 & 1000 \\ -0.2 & 1 & 780 \end{array}\right]$

$\xrightarrow{0.2R_1+R_2\to R_2} \left[\begin{array}{cc|c} 1 & -0.1 & 1000 \\ 0 & 0.98 & 980 \end{array}\right]$

$\xrightarrow{\frac{100}{98}R_2\to R_2} \left[\begin{array}{cc|c} 1 & -0.1 & 1000 \\ 0 & 1 & 1000 \end{array}\right]$

$\xrightarrow{0.1R_2+R_1\to R_1} \left[\begin{array}{cc|c} 1 & 0 & 1100 \\ 0 & 1 & 1000 \end{array}\right] \Rightarrow$

$a = 1100, b = 1000$

80. a. $\begin{cases} a = 0.5a + 0.2b + 50,000 \\ b = 0.6a + 0.3b + 40,000 \end{cases}$

b. Rewrite the system as

$\begin{cases} 0.5a - 0.2b = 50,000 \\ -0.6a + 0.7b = 40,000 \end{cases} \Rightarrow$

$\left[\begin{array}{cc|c} 0.5 & -0.2 & 50,000 \\ -0.6 & 0.7 & 40,000 \end{array}\right]$

c. $\left[\begin{array}{cc|c} 0.5 & -0.2 & 50,000 \\ -0.6 & 0.7 & 40,000 \end{array}\right]$

$\xrightarrow{2R_1\to R_1} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & 100,000 \\ -\frac{6}{10} & \frac{7}{10} & 40,000 \end{array}\right]$

$\xrightarrow{\frac{6}{10}R_1+R_2\to R_2} \left[\begin{array}{cc|c} 1 & -\frac{2}{5} & 100,000 \\ 0 & \frac{23}{50} & 100,000 \end{array}\right]$

$\xrightarrow[\frac{50}{23}R_2\to R_2]{\frac{20}{23}R_2+R_1\to R_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{4,300,000}{23} \\ 0 & 1 & \frac{5,000,000}{23} \end{array}\right] \Rightarrow$

$a = \frac{4,300,000}{23} \approx 186,957$

$b = \frac{5,000,000}{23} \approx 217,391$

81. a. $\begin{cases} l = 0.4t + 0.2f + 10,000 \\ t = 0.5l + 0.3t + 20,000 \\ f = 0.5l + 0.05t + 0.35f + 10,000 \end{cases}$

b. Rewrite the system as

$\begin{cases} l - 0.4t - 0.2f = 10,000 \\ -0.5l + 0.7t - 0f = 20,000 \\ -0.5l - 0.05t + 0.65f = 10,000 \end{cases} \Rightarrow$

$\left[\begin{array}{ccc|c} 1 & -0.4 & -0.2 & 10,000 \\ -0.5 & 0.7 & 0 & 20,000 \\ -0.5 & -0.05 & 0.65 & 10,000 \end{array}\right]$

c. $\left[\begin{array}{rrr|r} 1 & -0.4 & -0.2 & 10,000 \\ -0.5 & 0.7 & 0 & 20,000 \\ -0.5 & -0.05 & 0.65 & 10,000 \end{array}\right]$

$$\xrightarrow[R_1+2R_3\to R_3]{R_1+2R_2\to R_2}\left[\begin{array}{rrr|r} 1 & -0.4 & -0.2 & 10,000 \\ 0 & 1 & -0.2 & 50,000 \\ 0 & -0.5 & 1.1 & 30,000 \end{array}\right]$$

$$\xrightarrow{R_2+2R_3\to R_3}\left[\begin{array}{rrr|r} 1 & -0.4 & -0.2 & 10,000 \\ 0 & 1 & -0.2 & 50,000 \\ 0 & 0 & 2 & 110,000 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}R_3\to R_3]{R_1+0.4R_2\to R_1}\left[\begin{array}{rrr|r} 1 & 0 & -0.28 & 30,000 \\ 0 & 1 & -0.2 & 50,000 \\ 0 & 0 & 1 & 55,000 \end{array}\right]$$

$$\xrightarrow[R_2+0.2R_3\to R_2]{R_1+0.28R_3\to R_1}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 45,400 \\ 0 & 1 & 0 & 61,000 \\ 0 & 0 & 1 & 55,000 \end{array}\right]$$

$l = \$45,400,\ t = \$61,000,\ f = \$55,000$

82. a. $\begin{cases} a = 0.2b + 0.3c + 320 \\ b = 0.1a + 0.4c + 90 \\ c = 0.2a + 0.5b + 150 \end{cases}$

b. Rewrite the system as

$$\begin{cases} a - 0.2b - 0.3c = 320 \\ -0.1a + b - 0.4c = 90 \\ -0.2a - 0.5b + c = 150 \end{cases} \Rightarrow$$

$$\left[\begin{array}{rrr|r} 1 & -0.2 & -0.3 & 320 \\ -0.1 & 1 & -0.4 & 90 \\ -0.2 & -0.5 & 1 & 150 \end{array}\right]$$

c. $\left[\begin{array}{rrr|r} 1 & -0.2 & -0.3 & 320 \\ -0.1 & 1 & -0.4 & 90 \\ -0.2 & -0.5 & 1 & 150 \end{array}\right]$

$$\xrightarrow{-2R_2+R_3\to R_3}\left[\begin{array}{rrr|r} 1 & -0.2 & -0.3 & 320 \\ -0.1 & 1 & -0.4 & 90 \\ 0 & -2.5 & 1.8 & -30 \end{array}\right]$$

$$\xrightarrow{R_1+10R_2\to R_2}\left[\begin{array}{rrr|r} 1 & -0.2 & -0.3 & 320 \\ 0 & 9.8 & -4.3 & 1220 \\ 0 & -2.5 & 1.8 & -30 \end{array}\right]$$

$$\xrightarrow{R_2+\frac{9.8}{2.5}R_3\to R_3}\left[\begin{array}{rrr|r} 1 & -0.2 & -0.3 & 320 \\ 0 & 9.8 & -4.3 & 1220 \\ 0 & 0 & 2.756 & 1102.4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2.756}R_3\to R_3}\left[\begin{array}{rrr|r} 1 & -0.2 & -0.3 & 320 \\ 0 & 9.8 & -4.3 & 1220 \\ 0 & 0 & 1 & 400 \end{array}\right]$$

$$\xrightarrow{R_2+4.3R_3\to R_2}\left[\begin{array}{rrr|r} 1 & -0.2 & -0.3 & 320 \\ 0 & 9.8 & 0 & 2940 \\ 0 & 0 & 1 & 400 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9.8}R_2\to R_2}\left[\begin{array}{rrr|r} 1 & -0.2 & -0.3 & 320 \\ 0 & 1 & 0 & 300 \\ 0 & 0 & 1 & 400 \end{array}\right]$$

$$\xrightarrow{R_1+0.2R_2\to R_1}\left[\begin{array}{rrr|r} 1 & 0 & -0.3 & 380 \\ 0 & 1 & 0 & 300 \\ 0 & 0 & 1 & 400 \end{array}\right]$$

$$\xrightarrow{R_1+0.3R_3\to R_1}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 500 \\ 0 & 1 & 0 & 300 \\ 0 & 0 & 1 & 400 \end{array}\right]$$

$a = 500,\ b = 300,\ c = 400$

83. $T_1 = \dfrac{0+0+300+T_2}{4} \Rightarrow 4T_1 - T_2 = 300$

$T_2 = \dfrac{T_1+T_3+190+200}{4} \Rightarrow -T_1 + 4T_2 - T_3 = 390$

$T_3 = \dfrac{0+0+100+T_2}{4} \Rightarrow -T_2 + 4T_3 = 100$

$$\left[\begin{array}{rrr|r} 4 & -1 & 0 & 300 \\ -1 & 4 & -1 & 390 \\ 0 & -1 & 4 & 100 \end{array}\right]$$

$$\xrightarrow{R_1\leftrightarrow -R_2}\left[\begin{array}{rrr|r} 1 & -4 & 1 & -390 \\ 4 & -1 & 0 & 300 \\ 0 & -1 & 4 & 100 \end{array}\right]$$

$$\xrightarrow{-4R_1+R_2\to R_2}\left[\begin{array}{rrr|r} 1 & -4 & -1 & -390 \\ 0 & 15 & -4 & 1860 \\ 0 & -1 & 4 & 100 \end{array}\right]$$

$$\xrightarrow{R_2+R_3\to R_2}\left[\begin{array}{rrr|r} 1 & -4 & 1 & -390 \\ 0 & 14 & 0 & 1960 \\ 0 & -1 & 4 & 100 \end{array}\right]$$

$$\xrightarrow{\frac{1}{14}R_2\to R_2}\left[\begin{array}{rrr|r} 1 & -4 & 1 & -390 \\ 0 & 1 & 0 & 140 \\ 0 & -1 & 4 & 100 \end{array}\right]$$

$$\xrightarrow{R_2+R_3\to R_3}\left[\begin{array}{rrr|r} 1 & -4 & 1 & -390 \\ 0 & 1 & 0 & 140 \\ 0 & 0 & 4 & 240 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}R_3\to R_3]{R_1+4R_2\to R_1}\left[\begin{array}{rrr|r} 1 & 0 & 1 & 170 \\ 0 & 1 & 0 & 140 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

$$\xrightarrow{R_1-R_3\to R_1}\left[\begin{array}{rrr|r} 1 & 0 & 0 & 110 \\ 0 & 1 & 0 & 140 \\ 0 & 0 & 1 & 60 \end{array}\right] \Rightarrow$$

$T_1 = 110,\ T_2 = 140,\ T_3 = 60$

84. $T_1 = \dfrac{30 + T_2 + T_4 + 20}{4} \Rightarrow 4T_1 - T_2 - T_4 = 50$

$T_2 = \dfrac{T_1 + 30 + 50 + T_3}{4} \Rightarrow -T_1 + 4T_2 - T_3 = 80$

$T_3 = \dfrac{T_2 + 50 + 40 + T_4}{4} \Rightarrow -T_2 + 4T_3 - T_4 = 90$

$T_4 = \dfrac{T_1 + T_3 + 40 + 20}{4} \Rightarrow -T_1 - T_3 + 4T_4 = 60$

$$\left[\begin{array}{rrrr|r} 4 & -1 & 0 & -1 & 50 \\ -1 & 4 & -1 & 0 & 80 \\ 0 & -1 & 4 & -1 & 90 \\ -1 & 0 & -1 & 4 & 60 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow -R_2} \left[\begin{array}{rrrr|r} 1 & -4 & 1 & 0 & -80 \\ 4 & -1 & 0 & -1 & 50 \\ 0 & -1 & 4 & -1 & 90 \\ -1 & 0 & -1 & 4 & 60 \end{array}\right]$$

$$\xrightarrow[-4R_1 + R_2 \to R_2]{R_1 + R_4 \to R_4} \left[\begin{array}{rrrr|r} 1 & -4 & 1 & 0 & -80 \\ 0 & 15 & -4 & -1 & 370 \\ 0 & -1 & 4 & -1 & 90 \\ 0 & -4 & 0 & 4 & -20 \end{array}\right]$$

$$\xrightarrow{-4R_3 + R_4 \to R_4} \left[\begin{array}{rrrr|r} 1 & -4 & 1 & 0 & -80 \\ 0 & 15 & -4 & -1 & 370 \\ 0 & -1 & 4 & -1 & 90 \\ 0 & 0 & -16 & 8 & -380 \end{array}\right]$$

$$\xrightarrow[\frac{1}{8}R_4 \to R_4]{R_2 + 15R_3 \to R_3} \left[\begin{array}{rrrr|r} 1 & -4 & 1 & 0 & -80 \\ 0 & 15 & -4 & -1 & 370 \\ 0 & 0 & 56 & -16 & 1720 \\ 0 & 0 & -2 & 1 & -\frac{95}{2} \end{array}\right]$$

$$\xrightarrow[R_4 \to \frac{1}{12}R_4]{28R_4 + R_3 \to R_4} \left[\begin{array}{rrrr|r} 1 & -4 & 1 & 0 & -80 \\ 0 & 15 & -4 & -1 & 370 \\ 0 & 0 & 56 & -16 & 1720 \\ 0 & 0 & 0 & 1 & \frac{65}{2} \end{array}\right]$$

$$\xrightarrow[\frac{1}{56}R_3 \to R_3]{R_3 + 16R_4 \to R_3} \left[\begin{array}{rrrr|r} 1 & -4 & 1 & 0 & -80 \\ 0 & 15 & -4 & -1 & 370 \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & \frac{65}{2} \end{array}\right]$$

$$\xrightarrow[\substack{R_2 + R_4 \to R_2 \\ \frac{1}{15}R_2 \to R_2}]{R_2 + 4R_3 \to R_2} \left[\begin{array}{rrrr|r} 1 & -4 & 1 & 0 & -80 \\ 0 & 1 & 0 & 0 & \frac{75}{2} \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & \frac{65}{2} \end{array}\right]$$

$$\xrightarrow[R_1 - R_3 \to R_1]{R_1 + 4R_2 \to R_1} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 30 \\ 0 & 1 & 0 & 0 & \frac{75}{2} \\ 0 & 0 & 1 & 0 & 40 \\ 0 & 0 & 0 & 1 & \frac{65}{2} \end{array}\right]$$

$T_1 = 30,\ T_2 = \dfrac{75}{2},\ T_3 = 40,\ T_4 = \dfrac{65}{2}$

85. a. $\begin{cases} x - y = 270 - 200 \\ -x + z = 180 - 300 \\ y - z = 40 + 70 - 60 \end{cases}$

$\Rightarrow \begin{cases} x - y = 70 \\ -x + z = -120 \\ y - z = 50 \end{cases}$

b. $$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 70 \\ -1 & 0 & 1 & -120 \\ 0 & 1 & -1 & 50 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{rrr|r} 1 & -1 & 0 & 70 \\ 0 & -1 & 1 & -50 \\ 0 & 1 & -1 & 50 \end{array}\right]$$

$$\xrightarrow[R_2 + R_3 \to R_3]{R_1 - R_2 \to R_1} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 120 \\ 0 & -1 & 1 & -50 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$x - z = 120 \Rightarrow x = z + 120$

$-y + z = -50 \Rightarrow y = z + 50$

The solution is $\{(z + 120, z + 50, z)\}$.

There are 300 + 200 + 60 = 560 cars entering the system, so

$0 \le (z + 120) + (z + 50) + z \le 560$

$0 \le 3z + 170 \le 560$

$0 \le 3z \le 390$

$0 \le z \le 130$

Thus, the system has 131 solutions.

86. a. $\begin{cases} x - y = 450 - 350 \\ y - w = 350 - 550 \\ z - x = 370 - 350 \\ -z + w = 200 - 120 \end{cases} \Rightarrow$

$\begin{cases} x - y = 100 \\ y - w = -200 \\ -x + z = 20 \\ - z + w = 80 \end{cases}$

b. $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 100 \\ 0 & 1 & 0 & -1 & -200 \\ -1 & 0 & 1 & 0 & 20 \\ 0 & 0 & -1 & 1 & 80 \end{array}\right]$

$\xrightarrow{R_1+R_3\to R_3}\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 100 \\ 0 & 1 & 0 & 1 & -200 \\ 0 & -1 & 1 & 0 & 120 \\ 0 & 0 & -1 & 1 & 80 \end{array}\right]$

$\xrightarrow{R_2+R_3\to R_3}\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 100 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & -1 & -80 \\ 0 & 0 & -1 & 1 & 80 \end{array}\right]$

$\xrightarrow{R_3+R_4\to R_4}\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 100 \\ 0 & 1 & 0 & -1 & -200 \\ 0 & 0 & 1 & -1 & -80 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\Rightarrow$

$z-w=-80\Rightarrow z=w-80$
$y-w=-200\Rightarrow y=w-200$
$x-y=100\Rightarrow x-(w-200)=100\Rightarrow$
$x-w=-100\Rightarrow x=w-100$
The solution is $\{(w-100, w-200, w-80, w)\}$.

There are 350 + 550 + 120 + 350 = 1370 cars entering the system. So,

$$\begin{aligned}(w-100)+(w-200)+(w-80)+w &\le 1370\\ 4w-380 &\le 1370\\ 4w &\le 1750\\ w &\le 437.5\end{aligned}$$

From the diagram, we see that 200 cars leave the system through w, so $200\le w\le 437$. Thus, the system has 238 solutions.

87. $9=a(-1)^2+b(-1)+c=a-b+c$
$3=a(1)^2+b(1)+c=a+b+c$
$6=a(2)^2+b(2)+c=4a+2b+c$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 9 \\ 1 & 1 & 1 & 3 \\ 4 & 2 & 1 & 6 \end{array}\right]$

$\xrightarrow[-4R_1+R_3\to R_3]{-R_1+R_2\to R_2}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 9 \\ 0 & 2 & 0 & -6 \\ 0 & 6 & -3 & -30 \end{array}\right]$

$\xrightarrow[-\frac{1}{3}R_3\to R_3]{\frac{1}{2}R_2\to R_2}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & -2 & 1 & 10 \end{array}\right]$

$\xrightarrow{2R_2+R_3\to R_3}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

$\xrightarrow{R_1+R_2\to R_1}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

$\xrightarrow{R_1-R_3\to R_1}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

Thus, $a=2$, $b=-3$, and $c=4$. The equation is $y=2x^2-3x+4$.

88. a. $31=(0.5^2)a+0.5b+c=0.25a+0.5b+c$
$51=(1^2)a+b+c=a+b+c$
$67=2^2a+2b+c=4a+2b+c$

$\left[\begin{array}{ccc|c} 0.25 & 0.5 & 1 & 31 \\ 1 & 1 & 1 & 51 \\ 4 & 2 & 1 & 67 \end{array}\right]$

$\xrightarrow{4R_1\to R_1}\left[\begin{array}{ccc|c} 1 & 2 & 4 & 124 \\ 1 & 1 & 1 & 51 \\ 4 & 2 & 1 & 67 \end{array}\right]$

$\xrightarrow[-4R_1+R_3\to R_3]{R_1-R_2\to R_2}\left[\begin{array}{ccc|c} 1 & 2 & 4 & 124 \\ 0 & 1 & 3 & 73 \\ 0 & -6 & -15 & -429 \end{array}\right]$

$\xrightarrow{6R_2+R_3\to R_3}\left[\begin{array}{ccc|c} 1 & 2 & 4 & 124 \\ 0 & 1 & 3 & 73 \\ 0 & 0 & 3 & 9 \end{array}\right]$

$3c=9\Rightarrow c=3$
$b+3(3)=73\Rightarrow b=64$
$a+2(64)+4(3)=124\Rightarrow a=-16$
The equation is $h=-16t^2+64t+3$.

b. $-16t^2+64t+3=0\Rightarrow$

$$t=\frac{-64\pm\sqrt{64^2-4(-16)(3)}}{-32}=\frac{-64\pm\sqrt{4288}}{-32}\approx 4\text{ seconds}$$

c. The maximum occurs at $t=2$.
$h(2)=67$ feet.

9.1 Beyond the Basics

89. $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 2 & 4 & 0 \\ 2 & 0 & 1 & -1 & 0 \end{array}\right]$

$$\xrightarrow[-2R_1+R_3\to R_3]{R_1-R_2\to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 \\ 0 & -2 & -1 & -3 & 0 \end{array}\right]$$

$$\xrightarrow[R_2-R_3\to R_3]{\frac{1}{2}R_2+R_1\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2\to R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$y+\frac{1}{2}z+\frac{3}{2}w=0 \Rightarrow z=-2y-3w$
$x+\frac{1}{2}z-\frac{1}{2}w=0 \Rightarrow z=w-2x$
$-2y-3w=w-2x \Rightarrow x=2w+y$
The solution is $\left\{\left(y+2w,\ y,\ -2y-3w,\ w\right)\right\}$.

90. $\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 3 & 4 & 5 & 5 \\ 3 & 4 & 2 & -1 & 8 \end{array}\right]$

$$\xrightarrow[\substack{-2R_1+R_3\to R_3 \\ -3R_1+R_4\to R_4}]{-R_1+R_2\to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 3 & 0 & 2 & -2 \\ 0 & 5 & 2 & 7 & -3 \\ 0 & 7 & -1 & 2 & -4 \end{array}\right]$$

$$\xrightarrow[-7R_2+3R_4\to R_4]{-5R_2+3R_3\to R_3} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 3 & 0 & 2 & -2 \\ 0 & 0 & 6 & 11 & 1 \\ 0 & 0 & -3 & -8 & 2 \end{array}\right]$$

$$\xrightarrow[R_3+2R_4\to R_4]{\frac{1}{3}R_2\to R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 6 & 11 & 1 \\ 0 & 0 & 0 & -5 & 5 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{1}{6}R_3\to R_3 \\ -\frac{1}{5}R_4\to R_4}]{R_1+R_2\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -\frac{1}{3} & \frac{10}{3} \\ 0 & 1 & 0 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{11}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[\substack{-\frac{2}{3}R_4+R_2\to R_2 \\ -\frac{11}{6}R_4+R_3\to R_3}]{\frac{1}{3}R_4+R_1\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1-R_3\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

The solution is $\{(1, 0, 2, -1)\}$.

91. a. $\begin{bmatrix} 1 & 2 & -3 & 1 \\ 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 0 & -2 \end{bmatrix}$

$$\xrightarrow{R_1\leftrightarrow R_4} \begin{bmatrix} 2 & 3 & 0 & -2 \\ 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & -3 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1-2R_4\to R_4]{R_2-R_4\to R_2} \begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 6 & -4 \end{bmatrix}$$

$$\xrightarrow[R_2+2R_3\to R_3]{R_2-2R_4\to R_4} \begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -12 & 9 \end{bmatrix}$$

$$\xrightarrow{6R_3+R_4\to R_4} \begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

$$\xrightarrow[\substack{-\frac{1}{2}R_2\to R_2 \\ \frac{1}{2}R_3\to R_3 \\ \frac{1}{15}R_4\to R_4}]{\frac{1}{2}R_1\to R_1} \begin{bmatrix} 1 & \frac{3}{2} & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = B$$

(*continued on next page*)

(*continued*)

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 0 & -2 \end{bmatrix}$$

$$\xrightarrow[2R_1 - R_4 \to R_4]{R_1 - R_2 \to R_2} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -6 & 4 \end{bmatrix}$$

$$\xrightarrow{R_2 - 2R_4 \to R_4} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 12 & -9 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{12}R_4 \to R_4]{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} \end{bmatrix}$$

$$\xrightarrow{R_2 - R_3 \to R_3} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{4} \end{bmatrix}$$

$$\xrightarrow{R_3 + R_4 \to R_4} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{5}{4} \end{bmatrix}$$

$$\xrightarrow[-\frac{4}{5}R_4 \to R_4]{-R_3 \to R_3} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} = C$$

b. Use a calculator to show that the reduced row-echelon form of all three matrices is
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

92. a.
$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 6 \end{bmatrix}$$

$$\xrightarrow[R_1 - 2R_3 \to R_3]{R_1 - 2R_2 \to R_2} \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & -1 & -3 & -11 \end{bmatrix}$$

$$\xrightarrow{R_2 + 3R_3 \to R_3} \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & -12 & -36 \end{bmatrix}$$

$$\xrightarrow[\substack{\frac{1}{3}R_2 \to R_2 \\ -\frac{1}{12}R_3 \to R_3}]{\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 1 & 0.5 & -0.5 & 0.5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} = B$$

$z = 3$; $y - 3 = -1 \Rightarrow y = 2$,
$x + \frac{1}{2}(2) - \frac{1}{2}(3) = \frac{1}{2} \Rightarrow x = 1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[-2R_1 + R_3 \to R_3]{R_1 - R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & 0 & 4 \\ 0 & -1 & -3 & -11 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & -3 & -11 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_3 \to R_3} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} = C$$

$z = 3$; $y = 2$; $x + 2 + 3 = 6 \Rightarrow x = 1$
The solution is $\{(1, 2, 3)\}$.

b. Using a calculator, we find that the reduced row-echelon form of both matrices is
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

93. a. $\begin{bmatrix} a & b & m \\ c & d & n \end{bmatrix} \xrightarrow[\frac{1}{c}R_2 \to R_2]{\frac{1}{a}R_1 \to R_1} \begin{bmatrix} 1 & \frac{b}{a} & \frac{m}{a} \\ 1 & \frac{d}{c} & \frac{n}{c} \end{bmatrix}$

$\xrightarrow{R_1 - R_2 \to R_2} \begin{bmatrix} 1 & \frac{b}{a} & \frac{m}{a} \\ 0 & \frac{b}{a} - \frac{d}{c} & \frac{m}{a} - \frac{n}{c} \end{bmatrix}$

$= \begin{bmatrix} 1 & \frac{b}{a} & \frac{m}{a} \\ 0 & \frac{bc - ad}{ac} & \frac{cm - an}{ac} \end{bmatrix}$

$\xrightarrow{\frac{ac}{bc - da}R_2 \to R_2} \begin{bmatrix} 1 & \frac{b}{a} & \frac{m}{a} \\ 0 & 1 & \frac{cm - an}{bc - ad} \end{bmatrix}$

$\xrightarrow{-\frac{b}{a}R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 & \frac{dm - bn}{ad - bc} \\ 0 & 1 & \frac{cm - an}{bc - ad} \end{bmatrix}$

The solution is $\left(\frac{dm - bn}{ad - bc}, \frac{cm - an}{bc - ad} \right)$.

b. **(i)** There is a unique solution if $bc \ne ad$.

(ii) There is no solution if $bc = ad$ and $\frac{m}{b} \ne \frac{n}{d}$.

(iii) There are infinitely many solutions if $bc = ad$ and $\frac{m}{b} = \frac{n}{d}$.

94. Answers may vary.

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right], \left[\begin{array}{ccc|c} 2 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & -8 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 \end{array}\right]$

95. Using the hint suggested, we have

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & -1 & 3 & 10 \\ 5 & 5 & -4 & 3 \end{array}\right]$

$\xrightarrow[-5R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -4 & 0 & -8 \\ 0 & 0 & -9 & -27 \end{array}\right]$

$\xrightarrow[-\frac{1}{9}R_3 \to R_3]{-\frac{1}{4}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow$

$w = 3, v = 2, u + 2 + 3 = 6 \Rightarrow u = 1$
$\log x = 1 \Rightarrow x = 10;$
$\log y = 2 \Rightarrow y = 100$
$\log z = 3 \Rightarrow y = 1000$
The solution is $\{(10, 100, 1000)\}$.

96. Using the hint suggested, we have

$\left[\begin{array}{ccc|c} 4 & -3 & 1 & 1 \\ 1 & 4 & -2 & 10 \\ 2 & -2 & 3 & 4 \end{array}\right]$

$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 10 \\ 4 & -3 & 1 & 1 \\ 2 & -2 & 3 & 4 \end{array}\right]$

$\xrightarrow[-4R_1 + R_2 \to R_2]{R_2 - 2R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 10 \\ 0 & -19 & 9 & -39 \\ 0 & 1 & -5 & -7 \end{array}\right]$

$\xrightarrow{19R_3 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 10 \\ 0 & 0 & -86 & -172 \\ 0 & 1 & -5 & -7 \end{array}\right]$

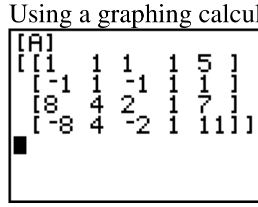
$\xrightarrow{-\frac{1}{86}R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 4 & -2 & 10 \\ 0 & 1 & -5 & -7 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow$

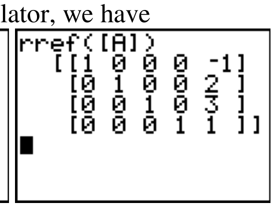
$w = 2, v - 5(2) = -7 \Rightarrow v = 3$
$u + 4(3) - 2(2) = 10 \Rightarrow u = 2$
$2 = 2^x \Rightarrow x = 1; \quad 3 = 3^y \Rightarrow y = 1$
$2 = 5^w \Rightarrow \ln 2 = w \ln 5 \Rightarrow w = \frac{\ln 2}{\ln 5}$
The solution is $\left\{ \left(1, 1, \frac{\ln 2}{\ln 5} \right) \right\}$.

97. $\begin{cases} a(1)^3 + b(1)^2 + c(1) + d = 5 \\ a(-1)^3 + b(-1)^2 + c(-1) + d = 1 \\ a(2)^3 + b(2)^2 + c(2) + d = 7 \\ a(-2)^3 + b(-2)^2 + c(-2) + d = 11 \end{cases} \Rightarrow$

$\begin{cases} a + b + c + d = 5 \\ -a + b - c + d = 1 \\ 8a + 4b + 2c + d = 7 \\ -8a + 4b - 2c + d = 11 \end{cases}$

Using a graphing calculator, we have





The equation is $y = -x^3 + 2x^2 + 3x + 1$.

98. $\begin{cases} a(1)^3+b(1)^2+c(1)+d=8 \\ a(-1)^3+b(-1)^2+c(-1)+d=2 \\ a(2)^3+b(2)^2+c(2)+d=8 \\ a(-2)^3+b(-2)^2+c(-2)+d=20 \end{cases} \Rightarrow$

$\begin{cases} a+b+c+d=8 \\ -a+b-c+d=2 \\ 8a+4b+2c+d=8 \\ -8a+4b-2c+d=20 \end{cases}$

Using a graphing calculator, we have

```
[A]
[[1  1  1  1  8 ]
 [-1 1  -1 1  2 ]
 [8  4  2  1  8 ]
 [-8 4  -2 1  20]]
```

```
rref([A])
  [[1 0 0 0 -2]
   [0 1 0 0 3 ]
   [0 0 1 0 5 ]
   [0 0 0 1 2 ]]
```

The equation is $y=-2x^3+3x^2+5x+2$.

9.1 Critical Thinking/Discussion/Writing

99. If $a \neq 0$, $\begin{bmatrix} a \\ b \end{bmatrix} \xrightarrow[\frac{1}{b}R_2 \to R_2]{\frac{1}{a}R_1 \to R_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{R_1 - R_2 \to R_2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

If $a = 0$, $b = 0$, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

100. If the matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then it is in reduced row-echelon form already. If the matrix is $\begin{bmatrix} 1 & k \\ 0 & 0 \end{bmatrix}$ (for k any real number), then it is in reduced row-echelon form already.

If the matrix is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then it is in reduced row-echelon form already. If the matrix is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\frac{1}{c}R_2 \to R_2]{\frac{1}{a}R_1 \to R_1} \begin{bmatrix} 1 & \frac{b}{a} \\ 1 & \frac{d}{c} \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2 \to R_2} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{b}{a} - \frac{d}{c} \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{bc-ad}{ac} \end{bmatrix}$$

$$\xrightarrow{\frac{ac}{bc-ad}R_2 \to R_2} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-\frac{b}{a}R_2 + R_1 \to R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

101. Yes. Use the inverse of the operation used to transform A to B.

102. a. True. For example,

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

b. False. For example,

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 6 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

9.1 Maintaining Skills

103. $3(x-1)=5-x \Rightarrow 3x-3=5-x \Rightarrow$
$4x=8 \Rightarrow x=2$
Solution set: $\{2\}$

104. $5(x-2)=3(x-3)+13 \Rightarrow$
$5x-10=3x-9+13 \Rightarrow 5x-10=3x+4 \Rightarrow$
$2x=14 \Rightarrow x=7$
Solution set: $\{7\}$

105. $-3(x+4)+2=8-x \Rightarrow -3x-12+2=8-x \Rightarrow$
$-3x-10=8-x \Rightarrow -2x=18 \Rightarrow x=-9$
Solution set: $\{-9\}$

106. $6x-3(5x+2)=4-5x \Rightarrow$
$6x-15x-6=4-5x \Rightarrow -9x-6=4-5x \Rightarrow$
$-4x=10 \Rightarrow x=-\frac{10}{4}=-\frac{5}{2}$
Solution set: $\left\{-\frac{5}{2}\right\}$

Be sure to check your solutions in the original equations in exercises 107–110.

107. $\begin{cases} 2x-y=5 & (1) \\ x+2y=25 & (2) \end{cases}$

From equation (1) we have $y = 2x - 5$.
Substituting this expression in equation (2) gives
$x+2(2x-5)=25 \Rightarrow x+4x-10=25 \Rightarrow$
$5x=35 \Rightarrow x=7$
Substitute $x = 7$ in equation (1) and solve for y.
$2(7)-y=5 \Rightarrow 14-y=5 \Rightarrow y=9$
Solution set: $\{(7, 9)\}$

108. $\begin{cases} x-3y=1 & (1) \\ 2x+y=-5 & (2) \end{cases}$

From equation (1) we have $x = 3y + 1$. Substituting this expression in equation (2) gives

$2(3y+1)+y=-5 \Rightarrow 6y+2+y=-5 \Rightarrow 7y=-7 \Rightarrow y=-1$

Substitute $y = -1$ in equation (1) and solve for x.

$x-3(-1)=1 \Rightarrow x+3=1 \Rightarrow x=-2$

Solution set: $\{(-2, -1)\}$

109. $\begin{cases} x+3y=6 & (1) \\ 2x+6y=8 & (2) \end{cases}$

Multiply equation (1) by -2, then add the two equations.

$$\begin{array}{r} -2x-6y=-12 \\ 2x+6y=8 \\ \hline 0=-4 \end{array} \text{ False}$$

Solution set: $\varnothing$

110. $\begin{cases} 2x-y=3 & (1) \\ 6x-3y=9 & (2) \end{cases}$

From equation (1) we have $y = 2x - 3$. Substituting this expression in equation (2) gives

$6x-3(2x-3)=9 \Rightarrow 6x-6x+9=9 \Rightarrow 0=0$

The system is dependent and the solution set can be expressed as $\{(x, 2x - 3)\}$.

111. True **112.** False

113. True. This is an example of the distributive property.

114. True. $\left(x^4+x^2\right)\left(\frac{10}{3}-3\right)=1 \Rightarrow \left(x^4+x^2\right)\left(\frac{1}{3}\right)=1 \Rightarrow x^4+x^2=3$

9.2 Matrix Algebra

9.2 Practice Problems

1. $\begin{bmatrix} 1 & 2x-y \\ x+y & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \Rightarrow \begin{cases} 2x-y=1 \\ x+y=2 \end{cases} \Rightarrow x=1,\ y=1$

2. $\begin{bmatrix} 2 & -1 & 4 \\ 5 & 0 & 9 \end{bmatrix} + \begin{bmatrix} -8 & 2 & 9 \\ 7 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 13 \\ 12 & 3 & 15 \end{bmatrix}$

3. $2A-3B = 2\begin{bmatrix} 7 & -4 \\ 3 & 6 \\ 0 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & -3 \\ 2 & 2 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 14 & -8 \\ 6 & 12 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 3 & -9 \\ 6 & 6 \\ 15 & 24 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 0 & 6 \\ -15 & -28 \end{bmatrix}$

4. $5\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} + 3X = 2\begin{bmatrix} 2 & 7 \\ -3 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & -5 \\ 15 & 25 \end{bmatrix} + 3X = \begin{bmatrix} 4 & 14 \\ -6 & -10 \end{bmatrix} \Rightarrow$

$3X = \begin{bmatrix} 4 & 14 \\ -6 & -10 \end{bmatrix} - \begin{bmatrix} 5 & -5 \\ 15 & 25 \end{bmatrix} = \begin{bmatrix} -1 & 19 \\ -21 & -35 \end{bmatrix} \Rightarrow X = \begin{bmatrix} -\frac{1}{3} & \frac{19}{3} \\ -7 & -\frac{35}{3} \end{bmatrix}$

5. Yes, the product matrix AB is defined. A is a 2×3 matrix, while B is a 3×1 matrix. The product matrix has order 2×1.

6. $N = \begin{bmatrix} S & C & M \end{bmatrix} = \begin{bmatrix} 10 & 30 & 45 \end{bmatrix};\ P = \begin{bmatrix} 41 \\ 26 \\ 19 \end{bmatrix}$

$$NP = \begin{bmatrix} 10 & 30 & 45 \end{bmatrix}\begin{bmatrix} 41 \\ 26 \\ 19 \end{bmatrix} = 10(41) + 30(26) + 45(19) = \$2045 \text{ thousand}$$

7. $\begin{bmatrix} 3 & -1 & 2 & 7 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3(-2) - 1(0) + 2(1) + 7(5) \end{bmatrix} = \begin{bmatrix} 31 \end{bmatrix}$

8. AB is not defined because A is a 2×2 matrix and B is a 3×2 matrix.

$$BA = \begin{bmatrix} 8 & 1 \\ -2 & 6 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 8(5) + 1(2) & 8(0) + 1(-1) \\ -2(5) + 6(2) & -2(0) + 6(-1) \\ 0(5) + 4(2) & 0(0) + 4(-1) \end{bmatrix} = \begin{bmatrix} 42 & -1 \\ 2 & -6 \\ 8 & -4 \end{bmatrix}$$

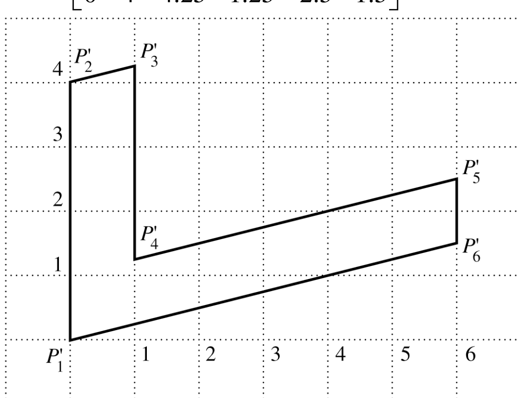
9. $AB = \begin{bmatrix} 7 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 7(2) + 1(4) & 7(-1) + 1(4) \\ 0(2) + 3(4) & 0(-1) + 3(4) \end{bmatrix} = \begin{bmatrix} 18 & -3 \\ 12 & 12 \end{bmatrix}$

$BA = \begin{bmatrix} 2 & -1 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 7 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2(7) - 1(0) & 2(1) - 1(3) \\ 4(7) + 4(0) & 4(1) + 4(3) \end{bmatrix} = \begin{bmatrix} 14 & -1 \\ 28 & 16 \end{bmatrix}$

10. $AD = \begin{bmatrix} 0 & 1 \\ 1 & 0.25 \end{bmatrix}\begin{bmatrix} 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 6 & 6 \end{bmatrix}$

$$= \begin{bmatrix} 0(0) + 1(0) & 0(4) + 1(0) & 0(4) + 1(1) & 0(1) + 1(1) & 0(1) + 1(6) & 0(0) + 1(6) \\ 1(0) + 0.25(0) & 1(4) + 0.25(0) & 1(4) + 0.25(1) & 1(1) + 0.25(1) & 1(1) + 0.25(6) & 1(0) + 0.25(6) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 & 6 & 6 \\ 0 & 4 & 4.25 & 1.25 & 2.5 & 1.5 \end{bmatrix}$$



9.2 Basic Concepts and Skills

1. Two $m \times n$ matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ are equal if $\underline{a_{ij} = b_{ij}}$ for all *i* and *j*.

2. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}, B = \begin{bmatrix} b_{ij} \end{bmatrix},$ and $C = \begin{bmatrix} c_{ij} \end{bmatrix}$. If $A + B = C$, then $c_{ij} = \underline{a_{ij} + b_{ij}}$ for all *i* and for all *j*.

3. The product of a $1 \times n$ matrix *A* and $n \times 1$ matrix *B* is $\underline{1 \times 1}$ matrix.

4. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then AB is defined and is an $\underline{m \times p}$ matrix.

5. False. It is possible that $AB = BA$, but not necessarily true.

6. False. The matrices must have the same order.

7. $\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ u \end{bmatrix} \Rightarrow x = 2, u = -3$

8. $\begin{bmatrix} -\frac{y}{2} \\ x \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \Rightarrow -\frac{y}{2} = 4 \Rightarrow x = 3,\ y = -8$

9. $\begin{bmatrix} 2 & x \\ y & -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -3 \end{bmatrix} \Rightarrow x = -1, y = 3$

10. $\begin{bmatrix} 2-x & 1 \\ -2 & 3+y \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & -3 \end{bmatrix} \Rightarrow$

$\begin{cases} 2 - x = 3 \Rightarrow x = -1 \\ 3 + y = -3 \Rightarrow y = -6 \end{cases}$

11. $\begin{bmatrix} 2x-3y & -4 \\ 5 & 3x+y \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 5 & 7 \end{bmatrix} \Rightarrow$

$\begin{cases} 2x - 3y = 1 \\ 3x + \ \ y = 7 \end{cases} \Rightarrow x = 2, y = 1$

12. $\begin{bmatrix} 3x+y & -17 \\ 19 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -17 \\ 19 & 2x+3y \end{bmatrix} \Rightarrow$

$\begin{cases} 3x + \ \ y = -\frac{1}{2} \\ 2x + 3y = 2 \end{cases} \Rightarrow x = -\frac{1}{2}, y = 1$

13. $\begin{bmatrix} x-y & 1 & 2 \\ 4 & 3x-2y & 3 \\ 5 & 6 & 5x-10y \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 4 & -1 & 3 \\ 5 & 6 & 6 \end{bmatrix} \Rightarrow$

$\begin{cases} x - \ \ y = -1 \\ 3x - \ 2y = -1 \\ 5x - 10y = 6 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 3 & -2 & -1 \\ 5 & -10 & 6 \end{array}\right]$

$\xrightarrow[-5R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & -5 & 11 \end{array}\right]$

$\xrightarrow{5R_2 + R_3 \to R_3} \left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 21 \end{array}\right] \Rightarrow 0 = 21$

There is no solution.

14. $\begin{bmatrix} x+y & 2 & 3 \\ 2 & x-y & 4 \\ 3 & 4 & 2x+3y \end{bmatrix} = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & -5 \end{bmatrix} \Rightarrow$

$\begin{cases} x + \ \ y = -1 \\ x - \ \ y = 5 \\ 2x + 3y = -5 \end{cases} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 1 & -1 & 5 \\ 2 & 3 & -5 \end{array}\right]$

$\xrightarrow[R_1 + R_2 \to R_1]{R_1 - R_2 \to R_2} \left[\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 2 & -6 \\ 2 & 3 & -5 \end{array}\right] \Rightarrow x = 2, y = -3$

15. a. $A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}$

b. $A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 7 \end{bmatrix}$

c. $-3A = -3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -9 & -12 \end{bmatrix}$

d. $3A - 2B = 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$

$= \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 4 & -6 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 5 & 18 \end{bmatrix}$

e. $(A + B)^2 = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}\right)^2$

$= \left(\begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}\right)^2 = \begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0(0) + 2(5) & 0(2) + 2(1) \\ 5(0) + 1(5) & 5(2) + 1(1) \end{bmatrix}$

$= \begin{bmatrix} 10 & 2 \\ 5 & 11 \end{bmatrix}$

f. $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 1(1) + 2(3) & 1(2) + 2(4) \\ 3(1) + 4(3) & 3(2) + 4(4) \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

$B^2 = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$

$= \begin{bmatrix} -1(-1) + 0(2) & -1(0) + 0(-3) \\ 2(-1) - 3(2) & 2(0) - 3(-3) \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ -8 & 9 \end{bmatrix}$

$A^2 - B^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -8 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 23 & 13 \end{bmatrix}$

16. a. $A+B=\begin{bmatrix}\frac{1}{3} & 1\\ -1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 2 & -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{4}{3} & 1\\ 1 & \frac{3}{2}\end{bmatrix}$

b. $A-B=\begin{bmatrix}\frac{1}{3} & 1\\ -1 & 2\end{bmatrix}-\begin{bmatrix}1 & 0\\ 2 & -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}-\frac{2}{3} & 1\\ -3 & \frac{5}{2}\end{bmatrix}$

c. $-3A=-3\begin{bmatrix}\frac{1}{3} & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-1 & -3\\ 3 & -6\end{bmatrix}$

d. $3A-2B=3\begin{bmatrix}\frac{1}{3} & 1\\ -1 & 2\end{bmatrix}-2\begin{bmatrix}1 & 0\\ 2 & -\frac{1}{2}\end{bmatrix}$
$=\begin{bmatrix}1 & 3\\ -3 & 6\end{bmatrix}-\begin{bmatrix}2 & 0\\ 4 & -1\end{bmatrix}=\begin{bmatrix}-1 & 3\\ -7 & 7\end{bmatrix}$

e. $(A+B)^2$
$=\left(\begin{bmatrix}\frac{1}{3} & 1\\ -1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 2 & -\frac{1}{2}\end{bmatrix}\right)^2=\left(\begin{bmatrix}\frac{4}{3} & 1\\ 1 & \frac{3}{2}\end{bmatrix}\right)^2$
$=\begin{bmatrix}\frac{4}{3} & 1\\ 1 & \frac{3}{2}\end{bmatrix}\begin{bmatrix}\frac{4}{3} & 1\\ 1 & \frac{3}{2}\end{bmatrix}$
$=\begin{bmatrix}\frac{4}{3}\left(\frac{4}{3}\right)+1(1) & \frac{4}{3}(1)+1\left(\frac{3}{2}\right)\\ 1\left(\frac{4}{3}\right)+\left(\frac{3}{2}\right)(1) & 1(1)+\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\end{bmatrix}$
$=\begin{bmatrix}\frac{25}{9} & \frac{17}{6}\\ \frac{17}{6} & \frac{13}{4}\end{bmatrix}$

f. $A^2=\begin{bmatrix}\frac{1}{3} & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{3} & 1\\ -1 & 2\end{bmatrix}$
$=\begin{bmatrix}\frac{1}{3}\left(\frac{1}{3}\right)+1(-1) & \frac{1}{3}(1)+1(2)\\ -1\left(\frac{1}{3}\right)+2(-1) & -1(1)+2(2)\end{bmatrix}$
$=\begin{bmatrix}-\frac{8}{9} & \frac{7}{3}\\ -\frac{7}{3} & 3\end{bmatrix}$

$B^2=\begin{bmatrix}1 & 0\\ 2 & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 2 & -\frac{1}{2}\end{bmatrix}$
$=\begin{bmatrix}1(1)+0(2) & 1(0)+0\left(-\frac{1}{2}\right)\\ 2(1)-\frac{1}{2}(2) & 2(0)-\frac{1}{2}\left(-\frac{1}{2}\right)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 1 & \frac{1}{4}\end{bmatrix}$

$A^2-B^2=\begin{bmatrix}-\frac{8}{9} & \frac{7}{3}\\ -\frac{7}{3} & 3\end{bmatrix}-\begin{bmatrix}1 & 0\\ 1 & \frac{1}{4}\end{bmatrix}=\begin{bmatrix}-\frac{17}{9} & \frac{7}{3}\\ -\frac{10}{3} & \frac{11}{4}\end{bmatrix}$

17. a. $A+B$ is not defined.

b. $A-B$ is not defined.

c. $-3A=-3\begin{bmatrix}2 & 3\\ -4 & 5\end{bmatrix}=\begin{bmatrix}-6 & -9\\ 12 & -15\end{bmatrix}$

d. $3A-2B$ is not defined

e. $(A+B)^2$ is not defined.

f. A^2-B^2 is not defined.

18. a. $A+B$ is not defined.

b. $A-B$ is not defined.

c. $-3A=-3\begin{bmatrix}1 & 2 & 3\\ -1 & -3 & 4\end{bmatrix}=\begin{bmatrix}-3 & -6 & -9\\ 3 & 9 & -12\end{bmatrix}$

d. $3A-2B$ is not defined

e. $(A+B)^2$ is not defined.

f. A^2-B^2 is not defined.

19. a. $A+B=\begin{bmatrix}4 & 0 & -1\\ -2 & 5 & 2\\ 0 & 0 & 1\end{bmatrix}+\begin{bmatrix}3 & 1 & 0\\ 1 & -4 & 2\\ 2 & 1 & 3\end{bmatrix}$
$=\begin{bmatrix}7 & 1 & -1\\ -1 & 1 & 4\\ 2 & 1 & 4\end{bmatrix}$

b. $A-B=\begin{bmatrix}4 & 0 & -1\\ -2 & 5 & 2\\ 0 & 0 & 1\end{bmatrix}-\begin{bmatrix}3 & 1 & 0\\ 1 & -4 & 2\\ 2 & 1 & 3\end{bmatrix}$
$=\begin{bmatrix}1 & -1 & -1\\ -3 & 9 & 0\\ -2 & -1 & -2\end{bmatrix}$

c. $-3A = -3\begin{bmatrix} 4 & 0 & -1 \\ -2 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -12 & 0 & 3 \\ 6 & -15 & -6 \\ 0 & 0 & -3 \end{bmatrix}$

d. $3A - 2B = 3\begin{bmatrix} 4 & 0 & -1 \\ -2 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 3 & 1 & 0 \\ 1 & -4 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 12 & 0 & -3 \\ -6 & 15 & 6 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 6 & 2 & 0 \\ 2 & -8 & 4 \\ 4 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ -8 & 23 & 2 \\ -4 & -2 & -3 \end{bmatrix}$

e. $(A+B)^2 = \begin{bmatrix} 7 & 1 & -1 \\ -1 & 1 & 4 \\ 2 & 1 & 4 \end{bmatrix}\begin{bmatrix} 7 & 1 & -1 \\ -1 & 1 & 4 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7(7)+1(-1)-1(2) & 7(1)+1(1)-1(1) & 7(-1)+1(4)-1(4) \\ -1(7)+1(-1)+4(2) & -1(1)+1(1)+4(1) & -1(-1)+1(4)+4(4) \\ 2(7)+1(-1)+4(2) & 2(1)+1(1)+4(1) & 2(-1)+1(4)+4(4) \end{bmatrix}$

$= \begin{bmatrix} 46 & 7 & -7 \\ 0 & 4 & 21 \\ 21 & 7 & 18 \end{bmatrix}$

f. $A^2 = \begin{bmatrix} 4 & 0 & -1 \\ -2 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & -1 \\ -2 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4(4)+0(-2)-1(0) & 4(0)+0(5)-1(0) & 4(-1)+0(2)-1(1) \\ -2(4)+5(-2)+2(0) & -2(0)+5(5)+2(0) & -2(-1)+5(2)+2(1) \\ 0(4)+0(-2)+1(0) & 0(0)+0(5)+1(0) & 0(-1)+0(2)+1(1) \end{bmatrix}$

$= \begin{bmatrix} 16 & 0 & -5 \\ -18 & 25 & 14 \\ 0 & 0 & 1 \end{bmatrix}$

$B^2 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -4 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -4 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3(3)+1(1)+0(2) & 3(1)+1(-4)+0(1) & 3(0)+1(2)+0(3) \\ 1(3)-4(1)+2(2) & 1(1)-4(-4)+2(1) & 1(0)-4(2)+2(3) \\ 2(3)+1(1)+3(2) & 2(1)+1(-4)+3(1) & 2(0)+1(2)+3(3) \end{bmatrix} = \begin{bmatrix} 10 & -1 & 2 \\ 3 & 19 & -2 \\ 13 & 1 & 11 \end{bmatrix}$

$A^2 - B^2 = \begin{bmatrix} 16 & 0 & -5 \\ -18 & 25 & 14 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 10 & -1 & 2 \\ 3 & 19 & -2 \\ 13 & 1 & 11 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -7 \\ -21 & 6 & 16 \\ -13 & -1 & -10 \end{bmatrix}$

20. a. $A + B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 4 \\ 2 & 3 & 1 \\ 1 & 5 & 2 \end{bmatrix}$

b. $A - B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix}$

c. $-3A = -3\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -6 \\ -6 & -3 & 0 \\ 0 & -3 & -9 \end{bmatrix}$

d. $3A - 2B = 3\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 0 & 3 & 9 \end{bmatrix} - \begin{bmatrix} 6 & -2 & 4 \\ 0 & 4 & 2 \\ 2 & 8 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 6 & -1 & -2 \\ -2 & -5 & 11 \end{bmatrix}$

e. $$(A+B)^2=\begin{bmatrix}4&-1&4\\2&3&1\\1&5&2\end{bmatrix}\begin{bmatrix}4&-1&4\\2&3&1\\1&5&2\end{bmatrix}=\begin{bmatrix}4(4)-1(2)+4(1)&4(-1)-1(3)+4(5)&4(4)-1(1)+4(2)\\2(4)+3(2)+1(1)&2(-1)+3(3)+1(5)&2(4)+3(1)+1(2)\\1(4)+5(2)+2(1)&1(-1)+5(3)+2(5)&1(4)+5(1)+2(2)\end{bmatrix}$$
$$=\begin{bmatrix}18&13&23\\15&12&13\\16&24&13\end{bmatrix}$$

f. $$A^2=\begin{bmatrix}1&0&2\\2&1&0\\0&1&3\end{bmatrix}\begin{bmatrix}1&0&2\\2&1&0\\0&1&3\end{bmatrix}=\begin{bmatrix}1(1)+0(2)+2(0)&1(0)+0(1)+2(1)&1(2)+0(0)+2(3)\\2(1)+1(2)+0(0)&2(0)+1(1)+0(1)&2(2)+1(0)+0(3)\\0(1)+1(2)+3(0)&0(0)+1(1)+3(1)&0(2)+1(0)+3(3)\end{bmatrix}=\begin{bmatrix}1&2&8\\4&1&4\\2&4&9\end{bmatrix}$$
$$B^2=\begin{bmatrix}3&-1&2\\0&2&1\\1&4&-1\end{bmatrix}\begin{bmatrix}3&-1&2\\0&2&1\\1&4&-1\end{bmatrix}=\begin{bmatrix}3(3)-1(0)+2(1)&3(-1)-1(2)+2(4)&3(2)-1(1)+2(-1)\\0(3)+2(0)+1(1)&0(-1)+2(2)+1(4)&0(2)+2(1)+1(-1)\\1(3)+4(0)-1(1)&1(-1)+4(2)-1(4)&1(2)+4(1)-1(-1)\end{bmatrix}$$
$$=\begin{bmatrix}11&3&3\\1&8&1\\2&3&7\end{bmatrix}$$
$$A^2-B^2=\begin{bmatrix}1&2&8\\4&1&4\\2&4&9\end{bmatrix}-\begin{bmatrix}11&3&3\\1&8&1\\2&3&7\end{bmatrix}=\begin{bmatrix}-10&-1&5\\3&-7&3\\0&1&2\end{bmatrix}$$

21. a. $$A+B=\begin{bmatrix}1&2&-3\\3&4&5\\2&-1&0\end{bmatrix}+\begin{bmatrix}3&1&0\\1&-4&2\\2&1&3\end{bmatrix}=\begin{bmatrix}4&3&-3\\4&0&7\\4&0&3\end{bmatrix}$$

b. $$A-B=\begin{bmatrix}1&2&-3\\3&4&5\\2&-1&0\end{bmatrix}-\begin{bmatrix}3&1&0\\1&-4&2\\2&1&3\end{bmatrix}=\begin{bmatrix}-2&1&-3\\2&8&3\\0&-2&-3\end{bmatrix}$$

c. $$-3A=-3\begin{bmatrix}1&2&-3\\3&4&5\\2&-1&0\end{bmatrix}=\begin{bmatrix}-3&-6&9\\-9&-12&-15\\-6&3&0\end{bmatrix}$$

d. $$3A-2B=3\begin{bmatrix}1&2&-3\\3&4&5\\2&-1&0\end{bmatrix}-2\begin{bmatrix}3&1&0\\1&-4&2\\2&1&3\end{bmatrix}=\begin{bmatrix}3&6&-9\\9&12&15\\6&-3&0\end{bmatrix}-\begin{bmatrix}6&2&0\\2&-8&4\\4&2&6\end{bmatrix}=\begin{bmatrix}-3&4&-9\\7&20&11\\2&-5&-6\end{bmatrix}$$

e. $$(A+B)^2=\begin{bmatrix}4&3&-3\\4&0&7\\4&0&3\end{bmatrix}\begin{bmatrix}4&3&-3\\4&0&7\\4&0&3\end{bmatrix}=\begin{bmatrix}4(4)+3(4)-3(4)&4(3)+3(0)-3(0)&4(-3)+3(7)-3(3)\\4(4)+0(4)+7(4)&4(3)+0(0)+7(0)&4(-3)+0(7)+7(3)\\4(4)+0(4)+3(4)&4(3)+0(0)+3(0)&4(-3)+0(7)+3(3)\end{bmatrix}$$
$$=\begin{bmatrix}16&12&0\\44&12&9\\28&12&-3\end{bmatrix}$$

f. $A^2 = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 4 & 5 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \\ 3 & 4 & 5 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1(1)+2(3)-3(2) & 1(2)+2(4)-3(-1) & 1(-3)+2(5)-3(0) \\ 3(1)+4(3)+5(2) & 3(2)+4(4)+5(-1) & 3(-3)+4(5)+5(0) \\ 2(1)-1(3)+0(2) & 2(2)-1(4)+0(-1) & 2(-3)-1(5)+0(0) \end{bmatrix}$

$= \begin{bmatrix} 1 & 13 & 7 \\ 25 & 17 & 11 \\ -1 & 0 & -11 \end{bmatrix}$

$B^2 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -4 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -4 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3(3)+1(1)+0(2) & 3(1)+1(-4)+0(1) & 3(0)+1(2)+0(3) \\ 1(3)-4(1)+2(2) & 1(1)-4(-4)+2(1) & 1(0)-4(2)+2(3) \\ 2(3)+1(1)+3(2) & 2(1)+1(-4)+3(1) & 2(0)+1(2)+3(3) \end{bmatrix} = \begin{bmatrix} 10 & -1 & 2 \\ 3 & 19 & -2 \\ 13 & 1 & 11 \end{bmatrix}$

$A^2 - B^2 = \begin{bmatrix} 1 & 13 & 7 \\ 25 & 17 & 11 \\ -1 & 0 & -11 \end{bmatrix} - \begin{bmatrix} 10 & -1 & 2 \\ 3 & 19 & -2 \\ 13 & 1 & 11 \end{bmatrix} = \begin{bmatrix} -9 & 14 & 5 \\ 22 & -2 & 13 \\ -14 & -1 & -22 \end{bmatrix}$

22. a. $A + B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 4 \\ 2 & 3 & 1 \\ 1 & 5 & 2 \end{bmatrix}$

b. $A - B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{bmatrix}$

c. $-3A = -3\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & -6 \\ -6 & -3 & 0 \\ 0 & -3 & -9 \end{bmatrix}$

d. $3A - 2B = 3\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} - 2\begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 \\ 6 & 3 & 0 \\ 0 & 3 & 9 \end{bmatrix} - \begin{bmatrix} 6 & -2 & 4 \\ 0 & 4 & 2 \\ 2 & 8 & -2 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 6 & -1 & -2 \\ -2 & -5 & 11 \end{bmatrix}$

e. $(A+B)^2 = \begin{bmatrix} 4 & -1 & 4 \\ 2 & 3 & 1 \\ 1 & 5 & 2 \end{bmatrix}\begin{bmatrix} 4 & -1 & 4 \\ 2 & 3 & 1 \\ 1 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 4(4)-1(2)+4(1) & 4(-1)-1(3)+4(5) & 4(4)-1(1)+4(2) \\ 2(4)+3(2)+1(1) & 2(-1)+3(3)+1(5) & 2(4)+3(1)+1(2) \\ 1(4)+5(2)+2(1) & 1(-1)+5(3)+2(5) & 1(4)+5(1)+2(2) \end{bmatrix}$

$= \begin{bmatrix} 18 & 13 & 23 \\ 15 & 12 & 13 \\ 16 & 24 & 13 \end{bmatrix}$

f. $A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)+0(2)+2(0) & 1(0)+0(1)+2(1) & 1(2)+0(0)+2(3) \\ 2(1)+1(2)+0(0) & 2(0)+1(1)+0(1) & 2(2)+1(0)+0(3) \\ 0(1)+1(2)+3(0) & 0(0)+1(1)+3(1) & 0(2)+1(0)+3(3) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 8 \\ 4 & 1 & 4 \\ 2 & 4 & 9 \end{bmatrix}$

$B^2 = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3(3)-1(0)+2(1) & 3(-1)-1(2)+2(4) & 3(2)-1(1)+2(-1) \\ 0(3)+2(0)+1(1) & 0(-1)+2(2)+1(4) & 0(2)+2(1)+1(-1) \\ 1(3)+4(0)-1(1) & 1(-1)+4(2)-1(4) & 1(2)+4(1)-1(-1) \end{bmatrix}$

$= \begin{bmatrix} 11 & 3 & 3 \\ 1 & 8 & 1 \\ 2 & 3 & 7 \end{bmatrix}$

$A^2 - B^2 = \begin{bmatrix} 1 & 2 & 8 \\ 4 & 1 & 4 \\ 2 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 11 & 3 & 3 \\ 1 & 8 & 1 \\ 2 & 3 & 7 \end{bmatrix} = \begin{bmatrix} -10 & -1 & 5 \\ 3 & -7 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} + X = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow$
$X = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix}$
$= \begin{bmatrix} -4 & -2 & 1 \\ 1 & 5 & 0 \end{bmatrix}$

24. $\begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} + X = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} \Rightarrow$
$X = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}$
$= \begin{bmatrix} 4 & 2 & -1 \\ -1 & -5 & 0 \end{bmatrix}$

25. $2X - \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow$
$2X = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} \Rightarrow$
$2X = \begin{bmatrix} 0 & 4 & -1 \\ 3 & 1 & 8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & 4 \end{bmatrix}$

26. $2X - \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} \Rightarrow$
$2X = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow$
$2X = \begin{bmatrix} 0 & 4 & -1 \\ 3 & 1 & 8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 0 & 2 & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & 4 \end{bmatrix}$

27. $2X + 3\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow$
$2X + \begin{bmatrix} 6 & 9 & -3 \\ 3 & -6 & 12 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow$
$2X = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 9 & -3 \\ 3 & -6 & 12 \end{bmatrix} \Rightarrow$
$2X = \begin{bmatrix} -8 & -8 & 3 \\ -1 & 9 & -8 \end{bmatrix} \Rightarrow X = \begin{bmatrix} -4 & -4 & \frac{3}{2} \\ -\frac{1}{2} & \frac{9}{2} & -4 \end{bmatrix}$

28. $3X + 2\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow$
$3X + \begin{bmatrix} 4 & 6 & -2 \\ 2 & -4 & 8 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow$
$3X = \begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 6 & -2 \\ 2 & -4 & 8 \end{bmatrix} \Rightarrow$
$3X = \begin{bmatrix} -6 & -5 & 2 \\ 0 & 7 & -4 \end{bmatrix} \Rightarrow X = \begin{bmatrix} -2 & -\frac{5}{3} & \frac{2}{3} \\ 0 & \frac{7}{3} & -\frac{4}{3} \end{bmatrix}$

29. $2\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} + 3\begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} + 4X = 0 \Rightarrow$
$\begin{bmatrix} 4 & 6 & -2 \\ 2 & -4 & 8 \end{bmatrix} + \begin{bmatrix} -6 & 3 & 0 \\ 6 & 9 & 12 \end{bmatrix} = -4X \Rightarrow$
$\begin{bmatrix} -2 & 9 & -2 \\ 8 & 5 & 20 \end{bmatrix} = -4X \Rightarrow X = \begin{bmatrix} \frac{1}{2} & -\frac{9}{4} & \frac{1}{2} \\ -2 & -\frac{5}{4} & -5 \end{bmatrix}$

30. $3X - 2\begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 4 \end{bmatrix} + 5\begin{bmatrix} -2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix} = 0 \Rightarrow$
$3X - \begin{bmatrix} 4 & 6 & -2 \\ 2 & -4 & 8 \end{bmatrix} + \begin{bmatrix} -10 & 5 & 0 \\ 10 & 15 & 20 \end{bmatrix} = 0 \Rightarrow$
$\begin{bmatrix} -14 & -1 & 2 \\ 8 & 19 & 12 \end{bmatrix} = -3X \Rightarrow$
$\begin{bmatrix} \frac{14}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{8}{3} & -\frac{19}{3} & -4 \end{bmatrix} = X$

31. a. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1(-2)+2(3) & 1(1)+2(5) \\ 3(-2)+4(3) & 3(1)+4(5) \end{bmatrix}$
$= \begin{bmatrix} 4 & 11 \\ 6 & 23 \end{bmatrix}$

b. $\begin{bmatrix} -2 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2(1)+1(3) & -2(2)+1(4) \\ 3(1)+5(3) & 3(2)+5(4) \end{bmatrix}$
$= \begin{bmatrix} 1 & 0 \\ 18 & 26 \end{bmatrix}$

32. a. $\begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 3(1)+2(2) & 3(3)+2(5) & 3(-2)+2(0) \\ 1(1)+5(2) & 1(3)+5(5) & 1(-2)+5(0) \\ 0(1)+1(2) & 0(3)+1(5) & 0(-2)+1(0) \end{bmatrix} = \begin{bmatrix} 7 & 19 & -6 \\ 11 & 28 & -2 \\ 2 & 5 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(3)+3(1)-2(0) & 1(2)+3(5)-2(1) \\ 2(3)+5(1)+0(0) & 2(2)+5(5)+0(1) \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ 11 & 29 \end{bmatrix}$

33. a. $\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 2(1)-1(-2)+0(4) & 2(5)-1(3)+0(0) \\ -3(1)+1(-2)+2(4) & -3(5)+1(3)+2(0) \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 3 & -12 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1(2)+5(-3) & 1(-1)+5(1) & 1(0)+5(2) \\ -2(2)+3(-3) & -2(-1)+3(1) & -2(0)+3(2) \\ 4(2)+0(-3) & 4(-1)+0(1) & 4(0)+0(2) \end{bmatrix} = \begin{bmatrix} -13 & 4 & 10 \\ -13 & 5 & 6 \\ 8 & -4 & 0 \end{bmatrix}$

34. a. The product *AB* is not defined.

b. The product *BA* is not defined.

35. a. $\begin{bmatrix} 2 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(1)+3(-2)+5(4) \end{bmatrix} = \begin{bmatrix} 16 \end{bmatrix}$

b. $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1(2) & 1(3) & 1(5) \\ -2(2) & -2(3) & -2(5) \\ 4(2) & 4(3) & 4(5) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 \\ -4 & -6 & -10 \\ 8 & 12 & 20 \end{bmatrix}$

36. a. $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\begin{bmatrix} -3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1(-3) & 1(0) & 1(2) \\ 2(-3) & 2(0) & 2(2) \\ -1(-3) & -1(0) & -1(2) \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ -6 & 0 & 4 \\ 3 & 0 & -2 \end{bmatrix}$

b. $\begin{bmatrix} -3 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3(1)+0(2)+2(-1) \end{bmatrix} = \begin{bmatrix} -5 \end{bmatrix}$

37. a. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1(1)+2(0)+3(2) & 1(2)+2(3)+3(0) & 1(-1)+2(1)+3(-3) \end{bmatrix} = \begin{bmatrix} 7 & 8 & -8 \end{bmatrix}$

b. The product *BA* is not defined.

38. a. The product *AB* is not defined.

b. $\begin{bmatrix} -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & -6 & 2 \\ 2 & 3 & 0 \\ 1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -2(4)+0(2)+1(1) & -2(-6)+0(3)+1(2) & -2(2)+0(0)+1(-3) \end{bmatrix} = \begin{bmatrix} -7 & 14 & -7 \end{bmatrix}$

39. a. $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 0 \\ 4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 2(3)+0(-1)+1(4) & 2(1)+0(2)+1(5) & 2(0)+0(0)+1(2) \\ 1(3)+4(-1)+2(4) & 1(1)+4(2)+2(5) & 1(0)+4(0)+2(2) \\ 3(3)-1(-1)+0(4) & 3(1)-1(2)+0(5) & 3(0)-1(0)+0(2) \end{bmatrix} = \begin{bmatrix} 10 & 7 & 2 \\ 7 & 19 & 4 \\ 10 & 1 & 0 \end{bmatrix}$

b. $\begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 0 \\ 4 & 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & 2 \\ 3 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 3(2)+1(1)+0(3) & 3(0)+1(4)+0(-1) & 3(1)+1(2)+0(0) \\ -1(2)+2(1)+0(3) & -1(0)+2(4)+0(-1) & -1(1)+2(2)+0(0) \\ 4(2)+5(1)+2(3) & 4(0)+5(4)+2(-1) & 4(1)+5(2)+2(0) \end{bmatrix} = \begin{bmatrix} 7 & 4 & 5 \\ 0 & 8 & 3 \\ 19 & 18 & 14 \end{bmatrix}$

40. a. $\begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & -3 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 & 0 \\ -1 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3(2)-1(-1)+2(3) & 3(-5)-1(2)+2(0) & 3(0)-1(-1)+2(2) \\ 0(2)+4(-1)-3(3) & 0(-5)+4(2)-3(0) & 0(0)+4(-1)-3(2) \\ 1(2)-2(-1)+2(3) & 1(-5)-2(2)+2(0) & 1(0)-2(-1)+2(2) \end{bmatrix}$

$= \begin{bmatrix} 13 & -17 & 5 \\ -13 & 8 & -10 \\ 10 & -9 & 6 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -5 & 0 \\ -1 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 0 & 4 & -3 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2(3)-5(0)+0(1) & 2(-1)-5(4)+0(-2) & 2(2)-5(-3)+0(2) \\ -1(3)+2(0)-1(1) & -1(-1)+2(4)-1(-2) & -1(2)+2(-3)-1(2) \\ 3(3)+0(0)+2(1) & 3(-1)+0(4)+2(-2) & 3(2)+0(-3)+2(2) \end{bmatrix}$

$= \begin{bmatrix} 6 & -22 & 19 \\ -4 & 11 & -10 \\ 11 & -7 & 10 \end{bmatrix}$

41. $AB = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 & 0 \\ 1 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3(2)+1(1)+2(3) & 3(5)+1(2)+2(0) & 3(0)+1(-1)+2(2) \\ 0(2)+4(1)+3(3) & 0(5)+4(2)+3(0) & 0(0)+4(-1)+3(2) \\ 1(2)-2(1)+2(3) & 1(5)-2(2)+2(0) & 1(0)-2(-1)+2(2) \end{bmatrix} = \begin{bmatrix} 13 & 17 & 3 \\ 13 & 8 & 2 \\ 6 & 1 & 6 \end{bmatrix}$

$BA = \begin{bmatrix} 2 & 5 & 0 \\ 1 & 2 & -1 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 3 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2(3)+5(0)+0(1) & 2(1)+5(4)+0(-2) & 2(2)+5(3)+0(2) \\ 1(3)+2(0)-1(1) & 1(1)+2(4)-1(-2) & 1(2)+2(3)-1(2) \\ 3(3)+0(0)+2(1) & 3(1)+0(4)+2(-2) & 3(2)+0(3)+2(2) \end{bmatrix} = \begin{bmatrix} 6 & 22 & 19 \\ 2 & 11 & 6 \\ 11 & -1 & 10 \end{bmatrix}$

So, $AB \neq BA$.

42. $AB = \begin{bmatrix} 5 & 2 & 4 \\ -6 & -3 & 10 \\ -2 & 6 & 5 \end{bmatrix}\begin{bmatrix} 4 & 1 & 2 \\ -3 & 0 & 5 \\ -1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 5(4)+2(-3)+4(-1) & 5(1)+2(0)+4(3) & 5(2)+2(5)+4(4) \\ -6(4)-3(-3)+10(-1) & -6(1)-3(0)+10(3) & -6(2)-3(5)+10(4) \\ -2(4)+6(-3)+5(-1) & -2(1)+6(0)+5(3) & -2(2)+6(5)+5(4) \end{bmatrix}$

$= \begin{bmatrix} 10 & 17 & 36 \\ -25 & 24 & 13 \\ -31 & 13 & 46 \end{bmatrix}$

$BA = \begin{bmatrix} 4 & 1 & 2 \\ -3 & 0 & 5 \\ -1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 & 4 \\ -6 & -3 & 10 \\ -2 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 4(5)+1(-6)+2(-2) & 4(2)+1(-3)+2(6) & 4(4)+1(10)+2(5) \\ -3(5)+0(-6)+5(-2) & -3(2)+0(-3)+5(6) & -3(4)+0(10)+5(5) \\ -1(5)+3(-6)+4(-2) & -1(2)+3(-3)+4(6) & -1(4)+3(10)+4(5) \end{bmatrix}$

$= \begin{bmatrix} 10 & 17 & 36 \\ -25 & 24 & 13 \\ -31 & 13 & 46 \end{bmatrix} \Rightarrow AB = BA$

43. $(AB)C = \left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}\right)\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix} = \begin{bmatrix}1(2)+2(3) & 1(-3)+2(5)\\3(2)+4(3) & 3(-3)+4(5)\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix} = \begin{bmatrix}8 & 7\\18 & 11\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix}$

$= \begin{bmatrix}8(0)+7(2) & 8(1)+7(4)\\18(0)+11(2) & 18(1)+11(4)\end{bmatrix} = \begin{bmatrix}14 & 36\\22 & 62\end{bmatrix}$

$A(BC) = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\left(\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2(0)-3(2) & 2(1)-3(4)\\3(0)+5(2) & 3(1)+5(4)\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}-6 & -10\\10 & 23\end{bmatrix}$

$= \begin{bmatrix}1(-6)+2(10) & 1(-10)+2(23)\\3(-6)+4(10) & 3(-10)+4(23)\end{bmatrix} = \begin{bmatrix}14 & 36\\22 & 62\end{bmatrix} \Rightarrow (AB)C = A(BC)$

44. $A(B+C) = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\left(\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}+\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & -2\\5 & 9\end{bmatrix} = \begin{bmatrix}1(2)+2(5) & 1(-2)+2(9)\\3(2)+4(5) & 3(-2)+4(9)\end{bmatrix} = \begin{bmatrix}12 & 16\\26 & 30\end{bmatrix}$

$AB + AC = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}+\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix} = \begin{bmatrix}1(2)+2(3) & 1(-3)+2(5)\\3(2)+4(3) & 3(-3)+4(5)\end{bmatrix}+\begin{bmatrix}1(0)+2(2) & 1(1)+2(4)\\3(0)+4(2) & 3(1)+4(4)\end{bmatrix}$

$= \begin{bmatrix}8 & 7\\18 & 11\end{bmatrix}+\begin{bmatrix}4 & 9\\8 & 19\end{bmatrix} = \begin{bmatrix}12 & 16\\26 & 30\end{bmatrix} \Rightarrow A(B+C) = AB + AC$

45. $(A+B)C = \left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}\right)\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix} = \begin{bmatrix}3 & -1\\6 & 9\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix} = \begin{bmatrix}3(0)-1(2) & 3(1)-1(4)\\6(0)+9(2) & 6(1)+9(4)\end{bmatrix} = \begin{bmatrix}-2 & -1\\18 & 42\end{bmatrix}$

$AC + BC = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix}+\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 4\end{bmatrix} = \begin{bmatrix}1(0)+2(2) & 1(1)+2(4)\\3(0)+4(2) & 3(1)+4(4)\end{bmatrix}+\begin{bmatrix}2(0)-3(2) & 2(1)-3(4)\\3(0)+5(2) & 3(1)+5(4)\end{bmatrix}$

$= \begin{bmatrix}4 & 9\\8 & 19\end{bmatrix}+\begin{bmatrix}-6 & -10\\10 & 23\end{bmatrix} = \begin{bmatrix}-2 & -1\\18 & 42\end{bmatrix} \Rightarrow (A+B)C = AC + BC$

46. $c(AB) = c\left(\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}\right) = c\begin{bmatrix}1(2)+2(3) & 1(-3)+2(5)\\3(2)+4(3) & 3(-3)+4(5)\end{bmatrix} = c\begin{bmatrix}8 & 7\\18 & 11\end{bmatrix} = \begin{bmatrix}8c & 7c\\18c & 11c\end{bmatrix}$

$(cA)B = \left(c\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\right)\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix} = \begin{bmatrix}c & 2c\\3c & 4c\end{bmatrix}\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix} = \begin{bmatrix}2c+6c & -3c+10c\\6c+12c & -9c+20c\end{bmatrix} = \begin{bmatrix}8c & 7c\\18c & 11c\end{bmatrix}$

$A(cB) = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\left(c\begin{bmatrix}2 & -3\\3 & 5\end{bmatrix}\right) = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}2c & -3c\\3c & 5c\end{bmatrix} = \begin{bmatrix}2c+6c & -3c+10c\\6c+12c & -9c+20c\end{bmatrix} = \begin{bmatrix}8c & 7c\\18c & 11c\end{bmatrix} \Rightarrow$

$c(AB) = (cA)B = A(cB)$

9.2 Applying the Concepts

47.

$$A + B = \begin{bmatrix}7 & 3 & 18\\4 & 1 & 3\end{bmatrix}+\begin{bmatrix}6 & 2 & 20\\3 & 1 & 4\end{bmatrix} = \begin{array}{c}\begin{array}{ccc}\text{Steel} & \text{Glass} & \text{Wood}\end{array}\\ \begin{bmatrix}13 & 5 & 38\\7 & 2 & 7\end{bmatrix}\end{array}\begin{array}{l}\text{Material}\\ \text{Transportation}\end{array}$$

48.

$$A + B = 3\begin{bmatrix}7 & 3 & 18\\4 & 1 & 3\end{bmatrix}+2\begin{bmatrix}6 & 2 & 20\\3 & 1 & 4\end{bmatrix} = \begin{bmatrix}21 & 9 & 54\\12 & 3 & 9\end{bmatrix}+\begin{bmatrix}12 & 4 & 40\\6 & 2 & 8\end{bmatrix} = \begin{array}{c}\begin{array}{ccc}\text{Steel} & \text{Glass} & \text{Wood}\end{array}\\ \begin{bmatrix}33 & 13 & 94\\18 & 5 & 17\end{bmatrix}\end{array}\begin{array}{l}\text{Material}\\ \text{Transportation}\end{array}$$

49. $\begin{bmatrix}100 & 300 & 400\end{bmatrix}\begin{bmatrix}60\\38\\17\end{bmatrix} = \begin{bmatrix}100(60)+300(38)+400(17)\end{bmatrix} = \begin{bmatrix}24,200\end{bmatrix}$

The total cost is $24,200.

50. $$\begin{bmatrix}50 & 75 & 150\end{bmatrix}\begin{array}{c}\begin{array}{ccc}\text{M} & \text{L} & \text{P}\end{array}\\ \begin{bmatrix}20 & 2 & 20\\ 15 & 3 & 25\\ 25 & 1 & 15\end{bmatrix}\end{array}\begin{array}{c}\\ \text{A}\\ \text{B}\\ \text{C}\end{array} = \begin{bmatrix}50(20)+75(15)+150(25) & 50(2)+75(3)+150(1) & 50(20)+75(25)+150(15)\end{bmatrix}$$

$$\begin{array}{c}\begin{array}{ccc}\text{Materials} & \text{Labor} & \text{Profit}\end{array}\\ =\begin{bmatrix}5875 & 475 & 5125\end{bmatrix}\end{array}$$

51. a.
$$\begin{array}{cc} & \begin{array}{ccc}\text{Chairman} & \text{President} & \text{Vice President}\end{array}\\ \begin{array}{c}\text{Salary}\\ \text{Bonus}\\ \text{Stock}\end{array} & \begin{bmatrix}2{,}500{,}000 & 1{,}250{,}000 & 100{,}000\\ 1{,}500{,}000 & 750{,}000 & 150{,}000\\ 50{,}000 & 25{,}000 & 5000\end{bmatrix}\end{array}$$

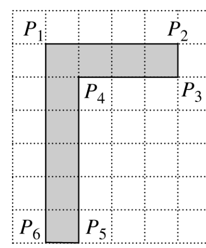
b. $$\begin{bmatrix}1\\ 1\\ 4\end{bmatrix}\begin{array}{l}\text{Chairman}\\ \text{President}\\ \text{Vice President}\end{array}$$

c. $$\begin{bmatrix}2{,}500{,}000 & 1{,}250{,}000 & 100{,}000\\ 1{,}500{,}000 & 750{,}000 & 150{,}000\\ 50{,}000 & 25{,}000 & 5000\end{bmatrix}\begin{bmatrix}1\\ 1\\ 4\end{bmatrix} = \begin{bmatrix}2{,}500{,}000(1)+1{,}250{,}000(1)+100{,}000(4)\\ 1{,}500{,}000(1)+750{,}000(1)+150{,}000(4)\\ 50{,}000(1)+25{,}000(1)+5000(4)\end{bmatrix} = \begin{array}{c}\text{Totals}\\ \begin{bmatrix}4{,}150{,}000\\ 2{,}850{,}000\\ 95{,}000\end{bmatrix}\end{array}\begin{array}{l}\\ \text{Salary}\\ \text{Bonus}\\ \text{Stock}\end{array}$$

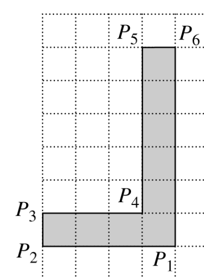
52. $$\begin{array}{cc} & \begin{array}{ccc}\text{M} & \text{F} & \text{C}\end{array}\\ \begin{array}{c}\text{Browns}\\ \text{Newgards}\end{array} & \begin{bmatrix}2 & 3 & 1\\ 1 & 1 & 2\end{bmatrix}\end{array}\begin{array}{c}\begin{array}{cc}\text{Cals} & \text{Protein}\end{array}\\ \begin{bmatrix}2400 & 55\\ 1900 & 45\\ 1800 & 33\end{bmatrix}\end{array} = \begin{bmatrix}2(2400)+3(1900)+1(1800) & 2(55)+3(45)+1(33)\\ 1(2400)+1(1900)+2(1800) & 1(55)+1(45)+2(33)\end{bmatrix} = \begin{array}{c}\begin{array}{cc}\text{Cals} & \text{Protein}\end{array}\\ \begin{bmatrix}12{,}300 & 278\\ 7900 & 166\end{bmatrix}\end{array}$$

For the Brown family, the total calorie requirement is 12,300 and the total protein requirement is 278. For the Newgard family, the total calorie requirement is 7900, and the total protein requirement is 166.

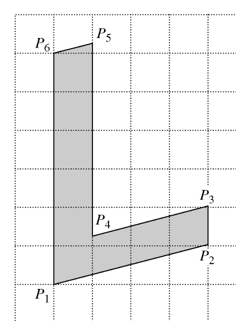
53. $$AD = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 4 & 4 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 6 & 6\end{bmatrix}$$
$$= \begin{bmatrix}1(0)+0(0) & 1(4)+0(0) & 1(4)+0(1) & 1(1)+0(1) & 1(1)+0(6) & 1(0)+0(6)\\ 0(0)-1(0) & 0(4)-1(0) & 0(4)-1(1) & 0(1)-1(1) & 0(1)-1(6) & 0(0)-1(6)\end{bmatrix}$$
$$= \begin{bmatrix}0 & 4 & 4 & 1 & 1 & 0\\ 0 & 0 & -1 & -1 & -6 & -6\end{bmatrix}$$



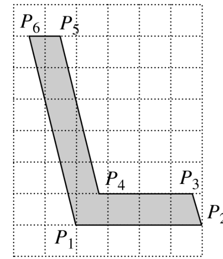
54. $$AD = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 4 & 4 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 6 & 6\end{bmatrix}$$
$$= \begin{bmatrix}-1(0)+0(0) & -1(4)+0(0) & -1(4)+0(1) & -1(1)+0(1) & -1(1)+0(6) & -1(0)+0(6)\\ 0(0)+1(0) & 0(4)+1(0) & 0(4)+1(1) & 0(1)+1(1) & 0(1)+1(6) & 0(0)+1(6)\end{bmatrix}$$
$$= \begin{bmatrix}0 & -4 & -4 & -1 & -1 & 0\\ 0 & 0 & 1 & 1 & 6 & 6\end{bmatrix}$$



55. $$AD = \begin{bmatrix}1 & 0\\ 0.25 & 1\end{bmatrix}\begin{bmatrix}0 & 4 & 4 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 6 & 6\end{bmatrix}$$
$$= \begin{bmatrix}1(0)+0(0) & 1(4)+0(0) & 1(4)+0(1) & 1(1)+0(1) & 1(1)+0(6) & 1(0)+0(6)\\ 0.25(0)+1(0) & 0.25(4)+1(0) & 0.25(4)+1(1) & 0.25(1)+1(1) & 0.25(1)+1(6) & 0.25(0)+1(6)\end{bmatrix}$$
$$= \begin{bmatrix}0 & 4 & 4 & 1 & 1 & 0\\ 0 & 1 & 2 & 1.25 & 6.25 & 6\end{bmatrix}$$



56. $$AD = \begin{bmatrix} 1 & -0.25 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 6 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 1(0)-0.25(0) & 1(4)-0.25(0) & 1(4)-0.25(1) & 1(1)-0.25(1) & 1(1)-0.25(6) & 1(0)-0.25(6) \\ 0(0)+1(0) & 0(4)+1(0) & 0(4)+1(1) & 0(1)+1(1) & 0(1)+1(6) & 0(0)+1(6) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 4 & 3.75 & 0.75 & -0.5 & -1.5 \\ 0 & 0 & 1 & 1 & 6 & 6 \end{bmatrix}$$



9.2 Beyond the Basics

57. $$AB = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0(2)+3(3) & 0(1)+3(0) \\ 0(2)+0(3) & 0(1)+0(0) \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix}$$
$$AC = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0(5)+3(3) & 0(4)+3(0) \\ 0(5)+0(3) & 0(4)+0(0) \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix}$$

So, $AB = AC$ does not imply $B = C$.

58. $$AB = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 2(-1)-3(1)-5(-1) & 2(3)-3(-3)-5(3) & 2(5)-3(-5)-5(5) \\ -1(-1)+4(1)+5(-1) & -1(3)+4(-3)+5(3) & -1(5)+4(-5)+5(5) \\ 1(-1)-3(1)-4(-1) & 1(3)-3(-3)-4(3) & 1(5)-3(-5)-4(5) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $AB = 0$ does not imply $A = 0$ or $B = 0$.

59. Answers will vary. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}$

$$(A+B)^2 = \begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 0(0)+2(5) & 0(2)+2(1) \\ 5(0)+1(5) & 5(2)+1(1) \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 5 & 11 \end{bmatrix}$$
$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1(1)+2(3) & 1(2)+2(4) \\ 3(1)+4(3) & 3(2)+4(4) \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$
$$2AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = 2\begin{bmatrix} 1(-1)+2(2) & 1(0)+2(-3) \\ 3(-1)+4(2) & 3(0)+4(-3) \end{bmatrix} = 2\begin{bmatrix} 3 & -6 \\ 5 & -12 \end{bmatrix} = \begin{bmatrix} 6 & -12 \\ 10 & -24 \end{bmatrix}$$
$$B^2 = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -1(-1)+0(2) & -1(0)+0(-3) \\ 2(-1)-3(2) & 2(0)-3(-3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 9 \end{bmatrix}$$
$$A^2 + 2AB + B^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} + \begin{bmatrix} 6 & -12 \\ 10 & -24 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -8 & 9 \end{bmatrix} = \begin{bmatrix} 14 & -2 \\ 17 & 7 \end{bmatrix}$$

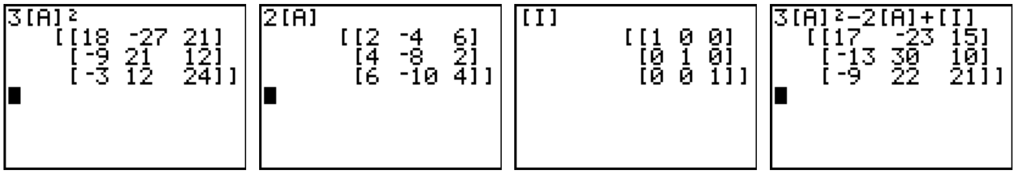
So, $(A+B)^2 \ne A^2 + 2AB + B^2$.

60. Using the matrices in exercise 59, we have $A^2 - B^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -8 & 9 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 23 & 13 \end{bmatrix}$.

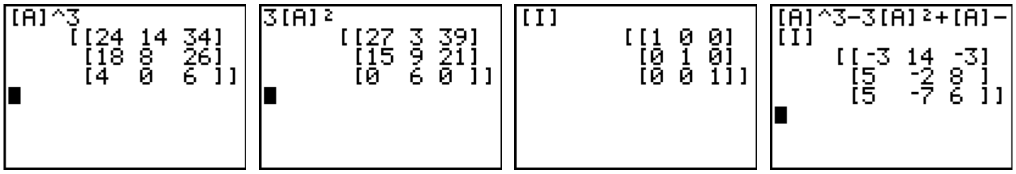
$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 7 \end{bmatrix} \qquad A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix}$$
$$(A-B)(A+B) = \begin{bmatrix} 2 & 2 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 2(0)+2(5) & 2(2)+2(1) \\ 1(0)+7(5) & 1(2)+7(1) \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 35 & 9 \end{bmatrix}$$

So, $A^2 - B^2 \ne (A-B)(A+B)$.

61.



62.



63. Let $B=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3z & 2y+3w \\ x+2z & y+2w \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} 2x+3z=1 \\ 2y+3w=0 \\ x+2z=0 \\ y+2w=1 \end{cases} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 3 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{array}\right] \xrightarrow[-(R_2-2R_4)\to R_4]{-(R_1-2R_3)\to R_3} \left[\begin{array}{cccc|c} 2 & 0 & 3 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow[\frac{1}{2}(R_2-3R_4)\to R_2]{\frac{1}{2}(R_1-3R_3)\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow$$

$$x=2,\ y=-3,\ z=-1,\ w=2 \Rightarrow B=\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

64. Let $B=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x-2z & y-2w \\ -2x+4z & -2y+4w \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} x-2z=1 \\ y-2w=0 \\ -2x+4z=0 \\ -2y+4w=1 \end{cases} \Rightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ -2 & 0 & 4 & 0 & 0 \\ 0 & -2 & 0 & 4 & 1 \end{array}\right] \xrightarrow[2R_2+R_4\to R_4]{2R_1+R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \text{There is no solution.}$$

65.

$$\left(4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}-\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \end{bmatrix}\right)\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \left(\begin{bmatrix} 8 & -4 & 12 \\ 4 & 0 & 8 \end{bmatrix}-\begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \end{bmatrix}\right)\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 7 & -6 & 13 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 7(2)-6(-1)+13(1) \\ 2(2)+3(-1)+4(1) \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} 33 \\ 5 \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow x=33,\ y=5$$

66. $\begin{bmatrix} 3 & 2 & -1 \\ 4 & 9 & 2 \\ 5 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x+2y-z \\ 4x+9y+2z \\ 5x-2z \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} 3x+2y- \quad z = 0 \\ 4x+9y+2z = 7 \Rightarrow \\ 5x- \qquad 2z = 2 \end{cases}$

$$\left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 4 & 9 & 2 & 7 \\ 5 & 0 & -2 & 2 \end{array}\right] \xrightarrow[-5R_1+3R_3 \to R_3]{-4R_1+3R_2 \to R_2} \left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 0 & 19 & 10 & 21 \\ 0 & -10 & -1 & 6 \end{array}\right] \xrightarrow{\frac{1}{81}(10R_2+19R_3) \to R_3} \left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 0 & 19 & 10 & 21 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow{\frac{1}{19}(R_2-10R_3) \to R_2} \left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_1+R_3 \to R_1} \left[\begin{array}{ccc|c} 3 & 2 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{\frac{1}{3}(R_1-2R_2) \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$x = 2, y = -1, z = 4$

67. $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0(1)+1(0) & 0(0)+1(0) \\ 0(1)+0(0) & 0(0)+0(0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1(0)+0(0) & 1(1)+0(0) \\ 0(0)+0(0) & 0(1)+0(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$

68. $A+B = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ $\qquad A-B = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

$(A+B)+(A-B) = 2A = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

$(A+B)-(A-B) = 2B = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

Verify by computing $A + B$ and $A - B$.

69. $2A-B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ $\qquad A+B = \begin{bmatrix} 3 & 0 & 3 \\ -2 & 1 & -4 \end{bmatrix}$

$(2A-B)+(A+B) = 3A = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 3 \\ -2 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 9 & -6 & 3 \\ -6 & 3 & -3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

$A+B = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix} + B = \begin{bmatrix} 3 & 0 & 3 \\ -2 & 1 & -4 \end{bmatrix} \Rightarrow B - \begin{bmatrix} 3 & 0 & 3 \\ -2 & 1 & -4 \end{bmatrix} - \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

70. $AB = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 2(-1)+(-3)(1)+(-5)(-1) & 2(3)+(-3)(-3)+(-5)(3) & 2(5)+(-3)(-5)+(-5)(5) \\ (-1)(-1)+4(1)+5(-1) & (-1)(3)+4(-3)+5(3) & (-1)(5)+4(-5)+5(5) \\ 1(-1)+(-3)(1)+(-4)(-1) & 1(3)+(-3)(-3)+(-4)(3) & 1(5)+(-3)(-5)+(-4)(5) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$BA = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$

$$= \begin{bmatrix} (-1)(2)+3(-1)+5(1) & (-1)(-3)+3(4)+5(-3) & (-1)(-5)+3(5)+5(-4) \\ 1(2)+(-3)(-1)+(-5)(1) & 1(-3)+(-3)(4)+(-5)(-3) & 1(5)+(-3)(-5)+(5)(-4) \\ (-1)(2)+3(-1)+5(1) & (-1)(-3)+3(4)+5(-3) & (-1)(-5)+3(5)+5(-4) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

71. $AB=\begin{bmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{bmatrix}\begin{bmatrix}2&-2&-4\\-1&3&4\\1&-2&-3\end{bmatrix}$

$=\begin{bmatrix}2(2)+(-3)(-1)+(-5)(1) & 2(-2)+(-3)(3)+(-5)(-2) & 2(-4)+(-3)(4)+(-5)(-3)\\(-1)(2)+4(-1)+5(1) & (-1)(-2)+4(3)+5(-2) & (-1)(-4)+4(4)+5(-3)\\1(2)+(-3)(-1)+(-4)(1) & 1(-2)+(-3)(3)+(-4)(-2) & 1(-4)+(-3)(4)+(-4)(-3)\end{bmatrix}$

$=\begin{bmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{bmatrix}=A$

$BA=\begin{bmatrix}2&-2&-4\\-1&3&4\\1&-2&-3\end{bmatrix}\begin{bmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{bmatrix}$

$=\begin{bmatrix}2(2)+(-2)(-1)+(-4)(1) & 2(-3)+(-2)(4)+(-4)(-3) & 2(-5)+(-2)(5)+(-4)(-4)\\(-1)(2)+3(-1)+4(1) & (-1)(-3)+3(4)+4(-3) & (-1)(-5)+3(5)+4(-4)\\1(2)+(-2)(-1)+(-3)(1) & 1(-3)+(-2)(4)+(-3)(-3) & 1(-5)+(-2)(5)+(-3)(-4)\end{bmatrix}$

$=\begin{bmatrix}2&-2&-4\\-1&3&4\\1&-2&-3\end{bmatrix}=B$

$A^2=\begin{bmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{bmatrix}\begin{bmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{bmatrix}$

$=\begin{bmatrix}2(2)+(-3)(-1)+(-5)(1) & 2(-3)+(-3)(4)+(-5)(-3) & 2(-5)+(-3)(5)+(-5)(-4)\\(-1)(2)+4(-1)+5(1) & (-1)(-3)+4(4)+5(-3) & (-1)(-5)+4(5)+5(-4)\\1(2)+(-3)(-1)+(-4)(1) & 1(-3)+(-3)(4)+(-4)(-3) & 1(-5)+(-3)(5)+(-4)(-4)\end{bmatrix}$

$=\begin{bmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{bmatrix}=A$

72. a. $A^2=\begin{bmatrix}2&3\\-1&2\end{bmatrix}\begin{bmatrix}2&3\\-1&2\end{bmatrix}=\begin{bmatrix}2(2)+3(-1) & 2(3)+3(2)\\(-1)(2)+2(-1) & (-1)(3)+2(2)\end{bmatrix}=\begin{bmatrix}1&12\\-4&1\end{bmatrix}$

$4A-7I=4\begin{bmatrix}2&3\\-1&2\end{bmatrix}-7\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}8&12\\-4&8\end{bmatrix}-\begin{bmatrix}7&0\\0&7\end{bmatrix}=\begin{bmatrix}1&12\\-4&1\end{bmatrix}=A^2$

b., c. $A^4=\left(A^2\right)^2=\begin{bmatrix}1&12\\-4&1\end{bmatrix}\begin{bmatrix}1&12\\-4&1\end{bmatrix}=\begin{bmatrix}1(1)+12(-4) & 1(12)+12(1)\\-4(1)+1(-4) & -4(12)+1(1)\end{bmatrix}=\begin{bmatrix}-47&24\\-8&-47\end{bmatrix}$

$8A-63I=8\begin{bmatrix}2&3\\-1&2\end{bmatrix}-63\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}16&24\\-8&16\end{bmatrix}-\begin{bmatrix}63&0\\0&63\end{bmatrix}=\begin{bmatrix}-47&24\\-8&-47\end{bmatrix}=A^4$

73. $A^2=\begin{bmatrix}4&-1&-4\\3&0&-4\\3&-1&-3\end{bmatrix}\begin{bmatrix}4&-1&-4\\3&0&-4\\3&-1&-3\end{bmatrix}$

$=\begin{bmatrix}4(4)+(-1)(3)+(-4)(3) & 4(-1)+(-1)(0)+(-4)(-1) & 4(-4)+(-1)(-4)+(-4)(-3)\\3(4)+0(3)+(-4)(3) & 3(-1)+0(0)+(-4)(-1) & 3(-4)+0(-4)+(-4)(-3)\\3(4)+(-1)(3)+(-3)(3) & 3(-1)+(-1)(0)+(-3)(-1) & 3(-4)+(-1)(-4)+(-3)(-3)\end{bmatrix}$

$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

9.2 Critical Thinking/Discussion/Writing

74.a. AB is defined when $n = 5$. The order of AB when this product is defined is $3 \times m$.

b. BA is defined when $m = 3$. The order of BA when this product is defined is $5 \times n$.

75. $(CA)B$

76.a. $$P^2 = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.3 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{bmatrix}\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.3 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.52 & 0.24 & 0.24 \\ 0.48 & 0.28 & 0.24 \\ 0.48 & 0.24 & 0.28 \end{bmatrix} = \begin{bmatrix} \frac{13}{25} & \frac{6}{25} & \frac{6}{25} \\ \frac{12}{25} & \frac{7}{25} & \frac{6}{25} \\ \frac{12}{25} & \frac{6}{25} & \frac{7}{25} \end{bmatrix}$$

$$XP^2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{13}{25} & \frac{6}{25} & \frac{6}{25} \\ \frac{12}{25} & \frac{7}{25} & \frac{6}{25} \\ \frac{12}{25} & \frac{6}{25} & \frac{7}{25} \end{bmatrix} = \begin{bmatrix} \frac{37}{75} & \frac{19}{75} & \frac{19}{75} \end{bmatrix}$$

b. $$P^3 = P^2P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.3 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{bmatrix}\begin{bmatrix} 0.52 & 0.24 & 0.24 \\ 0.48 & 0.28 & 0.24 \\ 0.48 & 0.24 & 0.28 \end{bmatrix} = \begin{bmatrix} 0.496 & 0.252 & 0.252 \\ 0.504 & 0.252 & 0.244 \\ 0.504 & 0.244 & 0.252 \end{bmatrix} = \begin{bmatrix} \frac{62}{125} & \frac{63}{250} & \frac{63}{250} \\ \frac{63}{125} & \frac{63}{250} & \frac{61}{250} \\ \frac{63}{125} & \frac{61}{250} & \frac{63}{250} \end{bmatrix}$$

$$XP^3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{62}{125} & \frac{63}{250} & \frac{63}{250} \\ \frac{63}{125} & \frac{63}{250} & \frac{61}{250} \\ \frac{63}{125} & \frac{61}{250} & \frac{63}{250} \end{bmatrix} = \begin{bmatrix} \frac{188}{375} & \frac{187}{750} & \frac{187}{750} \end{bmatrix}$$

$$P^4 = P^3P = \begin{bmatrix} 0.496 & 0.252 & 0.252 \\ 0.504 & 0.252 & 0.244 \\ 0.504 & 0.244 & 0.252 \end{bmatrix}\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.3 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.5008 & 0.2496 & 0.2496 \\ 0.4992 & 0.2512 & 0.2496 \\ 0.4992 & 0.2496 & 0.2512 \end{bmatrix} = \begin{bmatrix} \frac{313}{625} & \frac{156}{625} & \frac{156}{625} \\ \frac{312}{625} & \frac{157}{625} & \frac{156}{625} \\ \frac{312}{625} & \frac{156}{625} & \frac{157}{625} \end{bmatrix}$$

$$XP^4 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{313}{625} & \frac{156}{625} & \frac{156}{625} \\ \frac{312}{625} & \frac{157}{625} & \frac{156}{625} \\ \frac{312}{625} & \frac{156}{625} & \frac{157}{625} \end{bmatrix} = \begin{bmatrix} \frac{937}{1875} & \frac{469}{1875} & \frac{469}{1875} \end{bmatrix}$$

(*continued on next page*)

(*continued*)

$$P^5 = P^4P = \begin{bmatrix} 0.5008 & 0.2496 & 0.2496 \\ 0.4992 & 0.2512 & 0.2496 \\ 0.4992 & 0.2496 & 0.2512 \end{bmatrix}\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.3 & 0.1 \\ 0.6 & 0.1 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.49984 & 0.25008 & 0.25008 \\ 0.50016 & 0.25008 & 0.24976 \\ 0.50016 & 0.24976 & 0.25008 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1562}{3125} & \frac{1563}{6250} & \frac{1563}{6250} \\ \frac{1563}{3125} & \frac{1563}{6250} & \frac{1561}{6250} \\ \frac{1563}{3125} & \frac{1561}{6250} & \frac{1563}{6250} \end{bmatrix}$$

$$XP^5 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \frac{1562}{3125} & \frac{1563}{6250} & \frac{1563}{6250} \\ \frac{1563}{3125} & \frac{1563}{6250} & \frac{1561}{6250} \\ \frac{1563}{3125} & \frac{1561}{6250} & \frac{1563}{6250} \end{bmatrix} = \begin{bmatrix} \frac{4688}{9375} & \frac{4687}{18,750} & \frac{4687}{18,750} \end{bmatrix} \Rightarrow$$

$$XP^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

In the long term, the market shares for each company are: *A*, 50%; *B*, 25%; *C*, 25%.

9.2 Maintaining Skills

77. $\left(\frac{1}{2}\right)^{-1} = 2$

78. $\left(\frac{5}{12}\right)^{-1} = \frac{12}{5}$

79. $x^{-1} = \frac{1}{8} \Rightarrow x = 8$

80. $\left(\frac{x}{35}\right)^{-1} = 7 \Rightarrow \frac{35}{x} = 7 \Rightarrow 35 = 7x \Rightarrow x = 5$

81. $\frac{1}{4}x - \frac{7}{12} = \frac{11}{12} - \frac{5}{4}x \Rightarrow \frac{6}{4}x = \frac{18}{12} \Rightarrow$
$x = \frac{18}{12} \cdot \frac{4}{6} = 1$

82. $\frac{2}{3}x - \frac{3}{5} = \frac{4}{3}x + \frac{2}{15} \Rightarrow -\frac{2}{3}x = \frac{2}{15} + \frac{3}{5} \Rightarrow$
$-\frac{2}{3}x = \frac{11}{15} \Rightarrow x = \left(\frac{11}{15}\right)\left(-\frac{3}{2}\right) = -\frac{11}{10}$

83. $\frac{2}{4x-1} = \frac{3}{4x+1} \Rightarrow 2(4x+1) = 3(4x-1) \Rightarrow$
$8x + 2 = 12x - 3 \Rightarrow -4x = -5 \Rightarrow x = \frac{5}{4}$

84. $\frac{2}{x+1} = \frac{1}{x-1} \Rightarrow 2(x-1) = x+1 \Rightarrow$
$2x - 2 = x + 1 \Rightarrow x = 3$

85. $\begin{cases} 3x + 5y = 2 & (1) \\ 6x + 10y = 4 & (2) \end{cases}$

Multiply equation (1) by −2, then add the two equations.

$$\begin{array}{r} -6x - 10y = -4 \\ 6x + 10y = 4 \\ \hline 0 = 0 \end{array}$$

The equations are dependent and the system has infinitely many solutions. Solve equation (1) for *y* in terms of *x* to find the general solution.

$3x + 5y = 2 \Rightarrow 5y = -3x + 2 \Rightarrow$
$y = -\frac{3}{5}x + \frac{2}{5} = \frac{2-3x}{5}$

Thus, the solution set can be written as $\left\{\left(x, \frac{2-3x}{5}\right)\right\}$.

86. $\begin{cases} -9x + 2y = 8 & (1) \\ 5x + 8y = 32 & (2) \end{cases}$

Multiply equation (1) by −4, then add the equations and solve for *x*.

$$\begin{array}{r} 36x - 8y = -32 \\ 5x + 8y = 32 \\ \hline 41x = 0 \Rightarrow x = 0 \end{array}$$

Substitute $x = 0$ in equation (1) and solve for *y*.

$-9(0) + 2y = 8 \Rightarrow 2y = 8 \Rightarrow y = 4$

Solution set: {(0, 4)}

87. $\begin{cases} 3x+2y=6 & (1) \\ 6x+4y=-13 & (2) \end{cases}$

Multiply equation (1) by −2, then add the equations.

$$\begin{array}{r} -6x-4y=-12 \\ \underline{6x+4y=-13} \\ 0=-25 \end{array} \quad \text{False}$$

The system is inconsistent and the solution set is $\varnothing$.

88. $\begin{cases} 3x-4y=13 & (1) \\ 2x+5y=1 & (2) \end{cases}$

Multiply equation (1) by 5 and equation (2) by 4, then add the equations and solve for *x*.

$$\begin{array}{r} 15x-20y=65 \\ \underline{8x+20y=4} \\ 23x=69 \Rightarrow x=3 \end{array}$$

Substitute *x* = 3 in equation (2) and solve for *y*.

$2(3)+5y=1 \Rightarrow 6+5y=1 \Rightarrow 5y=-5 \Rightarrow y=-1$

Solution set: {(3, −1)}

9.3 The Matrix Inverse

9.3 Practice Problems

1. We need to verify that *AB* = *BA* = *I*.

$$AB=\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}=\begin{bmatrix} 3(-1)+2(2) & 3(2)+2(-3) \\ 2(-1)+1(2) & 2(2)+1(-3) \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA=\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} -1(3)+2(2) & -1(2)+2(1) \\ 2(3)-3(2) & 2(2)-3(1) \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus *B* is the inverse of *A*.

2. Suppose *A* has an inverse *B*, where $B=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then

$$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3x+z & 3y+w \\ 3x+z & 3y+w \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the matrices are equal, the entries must be equal. So $3x+z=1$ (in the 1, 1 position) and $3x+z=0$ (in the 2, 1) position. This is a contradiction, so *A* does not have an inverse.

3. $$[A\,|\,I]=\left[\begin{array}{rrr|rrr} 1 & 4 & -2 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 7 & -6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{rrr|rrr} 1 & 4 & -2 & 1 & 0 & 0 \\ 0 & 5 & 0 & 1 & 1 & 0 \\ 0 & -5 & 0 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3 \to R_3} \left[\begin{array}{rrr|rrr} 1 & 4 & -2 & 1 & 0 & 0 \\ 0 & 5 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$

The inverse of *A* does not exist.

4. $[A\,|\,I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 4 & 5 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-4R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 7 & 7 & 2 & 1 & 0 \\ 0 & -3 & -14 & -4 & 0 & 1 \end{array}\right]$

$$\xrightarrow{\frac{1}{11}(2R_2+R_3)\to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{11} & \frac{1}{11} \\ 0 & -3 & -14 & -4 & 0 & 1 \end{array}\right] \xrightarrow[\left(-\frac{1}{14}\right)(3R_2+R_3)\to R_3]{R_1-2R_2\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & -\frac{4}{11} & -\frac{2}{11} \\ 0 & 1 & 0 & 0 & \frac{2}{11} & \frac{1}{11} \\ 0 & 0 & 1 & \frac{2}{7} & -\frac{3}{77} & -\frac{1}{11} \end{array}\right]$$

$$\xrightarrow{R_1-3R_3\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & \frac{1}{7} & -\frac{19}{77} & \frac{1}{11} \\ 0 & 1 & 0 & 0 & \frac{2}{11} & \frac{1}{11} \\ 0 & 0 & 1 & \frac{2}{7} & -\frac{3}{77} & -\frac{1}{11} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{7} & -\frac{19}{77} & \frac{1}{11} \\ 0 & \frac{2}{11} & \frac{1}{11} \\ \frac{2}{7} & -\frac{3}{77} & -\frac{1}{11} \end{bmatrix}$$

5. a. $ad - bc = (8)(1) - (2)(4) = 0$

The inverse of matrix A does not exist.

b. $ad - bc = (8)(1) - (-2)(3) = 14$

Thus, matrix B is invertible.

$$B^{-1} = \frac{1}{14}\begin{bmatrix} 1 & 2 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} \frac{1}{14} & \frac{1}{7} \\ -\frac{3}{14} & \frac{4}{7} \end{bmatrix}$$

6. $\begin{cases} 3x + 2y + 3z = 9 \\ 3x + y = 12 \\ x + z = 6 \end{cases} \Rightarrow AX = B \Rightarrow \begin{bmatrix} 3 & 2 & 3 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 6 \end{bmatrix} \Rightarrow X = A^{-1}B$

Using a graphing calculator, we find that $A^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{3}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{bmatrix}$.

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{1}{5} & -\frac{6}{5} \\ \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 9 \\ 12 \\ 6 \end{bmatrix} \Rightarrow X = \begin{bmatrix} \frac{11}{2} \\ -\frac{9}{2} \\ \frac{1}{2} \end{bmatrix}$$

Solution set: $\left\{\left(\frac{11}{2}, -\frac{9}{2}, \frac{1}{2}\right)\right\}$

7. Following the discussion given in Example 7, we have $X=(I-A)^{-1}D=\frac{1}{19}\begin{bmatrix}36 & 24\\ 16 & 36\end{bmatrix}\begin{bmatrix}800\\ 3400\end{bmatrix}=\begin{bmatrix}\frac{110,400}{19}\\ \frac{135,200}{19}\end{bmatrix}$

To meet the consumer demand for 800 units of energy and 3400 units of food, the energy produced must be $\frac{110,400}{19}$ units and food production must be $\frac{135,200}{19}$ units.

8. Associate each letter in the phrase with the number representing its position in the alphabet, and partition the numbers into groups of three (inserting two zeros at the right end), forming a 3 × 6 matrix:

J	A	C	K		I	S		N	O	W		S	A	F	E		
[10	1	3]	[11	0	9]	[19	0	14]	[15	23	0]	[19	1	6]	[5	0	0]

$$M=\begin{bmatrix}10 & 11 & 19 & 15 & 19 & 5\\ 1 & 0 & 0 & 23 & 1 & 0\\ 3 & 9 & 14 & 0 & 6 & 0\end{bmatrix}$$

Let $A=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{bmatrix}$. Then, $AM=\begin{bmatrix}1 & 2 & 3\\ 1 & 3 & 3\\ 1 & 2 & 4\end{bmatrix}\begin{bmatrix}10 & 11 & 19 & 15 & 19 & 5\\ 1 & 0 & 0 & 23 & 1 & 0\\ 3 & 9 & 14 & 0 & 6 & 0\end{bmatrix}=\begin{bmatrix}21 & 38 & 61 & 61 & 39 & 5\\ 22 & 38 & 61 & 84 & 40 & 5\\ 24 & 47 & 75 & 61 & 45 & 5\end{bmatrix}$

The cryptogram is 21 22 24 38 38 47 61 61 75 61 84 61 39 40 45 5 5 5.

9.3 Basic Concepts and Skills

1. For $n\times n$ matrices A and B, if $AB = I$, then B is called the <u>inverse</u> of A.

2. An $n\times n$ matrix is invertible if there is a matrix B such that $\underline{AB = BA} = I$.

3. To find the inverse of an invertible matrix A, we transform $[A\,|\,I]$ by a sequence of row operations into $[I\,|\,B]$, where $B = \underline{A^{-1}}$.

4. $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is invertible if and only if $\underline{ad-bc\neq 0}$.

5. False.

6. False. $ad-bc=8(8)-16(4)=0$, so A is not invertible.

7. $AB=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$BA=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Rightarrow$

B is the inverse of A.

8. $AB=\begin{bmatrix}3 & 2\\ 4 & 3\end{bmatrix}\begin{bmatrix}3 & -2\\ -4 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$BA=\begin{bmatrix}3 & -2\\ -4 & 3\end{bmatrix}\begin{bmatrix}3 & 2\\ 4 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Rightarrow$

B is the inverse of A.

9. $AB=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}\begin{bmatrix}\frac{2}{5} & -\frac{1}{5}\\ -\frac{1}{10} & \frac{3}{10}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$BA=\begin{bmatrix}\frac{2}{5} & -\frac{1}{5}\\ -\frac{1}{10} & \frac{3}{10}\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Rightarrow$

B is the inverse of A.

10. $AB=\begin{bmatrix}2 & -3\\ 4 & -3\end{bmatrix}\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ -\frac{2}{3} & \frac{1}{3}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$BA=\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ -\frac{2}{3} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}2 & -3\\ 4 & -3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Rightarrow$

B is the inverse of A.

11. $AB=\begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$BA=\begin{bmatrix}2 & 1\\ 1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1\\ -1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 0\\ 0 & 1 & 0\end{bmatrix}\Rightarrow$

B is not the inverse of A.

12. $AB = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}\left(\frac{1}{8}\begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$BA = \left(\frac{1}{8}\begin{bmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{bmatrix}\right)\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow B$ is the inverse of A.

13. $AB = \begin{bmatrix} 1 & -2 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & -1 & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow B$ is not the inverse of A.

14. $AB = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\left(\frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$BA = \left(\frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}\right)\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow B$ is the inverse of A.

15. $AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & -1 \end{bmatrix}\left(\frac{1}{4}\begin{bmatrix} 5 & -2 & -1 \\ 2 & 0 & 2 \\ -3 & 2 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$BA = \left(\frac{1}{4}\begin{bmatrix} 5 & -2 & -1 \\ 2 & 0 & 2 \\ -3 & 2 & -1 \end{bmatrix}\right)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow B$ is the inverse of A.

16. $AB = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$BA = \left(\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow B$ is the inverse of A.

In exercises 17–26, the steps to find the inverse may vary. Be sure to verify that $AA^{-1} = I$.

17. $[A \mid I] = \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_1 - R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & -3 & \frac{1}{2} & -1 \end{array}\right]$

$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{6} & \frac{1}{3} \end{array}\right] \Rightarrow$

$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$

18. $[A\,|\,I]=\left[\begin{array}{cc|cc}4&3&1&0\\1&0&0&1\end{array}\right]$

$\xrightarrow{R_1-4R_2\to R_2}\left[\begin{array}{cc|cc}4&3&1&0\\0&3&1&-4\end{array}\right]\xrightarrow{\frac{1}{3}R_2}\left[\begin{array}{cc|cc}4&3&1&0\\0&1&\frac{1}{3}&-\frac{4}{3}\end{array}\right]$

$\xrightarrow{R_1-3R_2\to R_1}\left[\begin{array}{cc|cc}4&0&0&4\\0&1&\frac{1}{3}&-\frac{4}{3}\end{array}\right]\xrightarrow{\frac{1}{4}R_1}\left[\begin{array}{cc|cc}1&0&0&1\\0&1&\frac{1}{3}&-\frac{4}{3}\end{array}\right]\Rightarrow$

$A^{-1}=\begin{bmatrix}0&1\\\frac{1}{3}&-\frac{4}{3}\end{bmatrix}$

19. $[A\,|\,I]=\left[\begin{array}{cc|cc}2&4&1&0\\3&6&0&1\end{array}\right]$

$\xrightarrow{\frac{1}{2}R_1\to R_1}\left[\begin{array}{cc|cc}1&2&\frac{1}{2}&0\\3&6&0&1\end{array}\right]\xrightarrow{-3R_1+R_2\to R_2}\left[\begin{array}{cc|cc}1&2&\frac{1}{2}&0\\0&0&-\frac{3}{2}&1\end{array}\right]\Rightarrow$

The inverse of A does not exist.

20. $[A\,|\,I]=\left[\begin{array}{cc|cc}9&6&1&0\\6&4&0&1\end{array}\right]$

$\xrightarrow[\frac{1}{6}R_2\to R_2]{\frac{1}{9}R_1\to R_1}\left[\begin{array}{cc|cc}1&\frac{2}{3}&\frac{1}{9}&0\\1&\frac{2}{3}&0&\frac{1}{6}\end{array}\right]\xrightarrow{R_1-R_2\to R_2}\left[\begin{array}{cc|cc}1&\frac{2}{3}&\frac{1}{9}&0\\0&0&\frac{1}{9}&-\frac{1}{6}\end{array}\right]$

The inverse of A does not exist.

21. $[A\,|\,I]=\left[\begin{array}{ccc|ccc}1&6&4&1&0&0\\0&2&3&0&1&0\\0&1&2&0&0&1\end{array}\right]$

$\xrightarrow{2R_3-R_2\to R_3}\left[\begin{array}{ccc|ccc}1&6&4&1&0&0\\0&2&3&0&1&0\\0&0&1&0&-1&2\end{array}\right]$

$\xrightarrow{\frac{1}{2}(-3R_3+R_2)\to R_2}\left[\begin{array}{ccc|ccc}1&6&4&1&0&0\\0&1&0&0&2&-3\\0&0&1&0&-1&2\end{array}\right]$

$\xrightarrow[R_1-6R_2\to R_1]{R_1-4R_3\to R_1}\left[\begin{array}{ccc|ccc}1&0&0&1&-8&10\\0&1&0&0&2&-3\\0&0&1&0&-1&2\end{array}\right]$

$A^{-1}=\begin{bmatrix}1&-8&10\\0&2&-3\\0&-1&2\end{bmatrix}$

22. $[A\,|\,I]=\left[\begin{array}{ccc|ccc}2&0&-1&1&0&0\\5&1&0&0&1&0\\0&1&3&0&0&1\end{array}\right]$

$\xrightarrow[R_2-R_3\to R_2]{R_1+3R_3\to R_1}\left[\begin{array}{ccc|ccc}2&0&-1&3&0&-1\\5&0&-3&0&1&-1\\0&1&3&3&0&1\end{array}\right]$

$\xrightarrow{-3R_1+R_2\to R_1}\left[\begin{array}{ccc|ccc}-1&0&0&-3&1&-1\\5&0&-3&0&1&-1\\0&1&3&0&0&1\end{array}\right]$

$\xrightarrow{-R_1\to R_1}\left[\begin{array}{ccc|ccc}1&0&0&3&-1&1\\5&0&-3&0&1&-1\\0&1&3&0&0&1\end{array}\right]$

$\xrightarrow{-\frac{1}{3}(R_2-5R_1)\to R_2}\left[\begin{array}{ccc|ccc}1&0&0&3&-1&1\\0&0&1&5&-2&2\\0&1&3&0&0&1\end{array}\right]$

$\xrightarrow{R_2\leftrightarrow R_3}\left[\begin{array}{ccc|ccc}1&0&0&3&-1&1\\0&1&3&0&0&1\\0&0&1&5&-2&2\end{array}\right]$

$\xrightarrow{R_2-3R_3\to R_2}\left[\begin{array}{ccc|ccc}1&0&0&3&-1&1\\0&1&0&-15&6&-5\\0&0&1&5&-2&2\end{array}\right]$

$A^{-1}=\begin{bmatrix}3&-1&1\\-15&6&-5\\5&-2&2\end{bmatrix}$

23. $[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 3 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_2 - R_1 \to R_3} \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_1 - R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - 7R_2 \to R_3]{R_1 + R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 0 & 2 & 7 & -7 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}(R_3 - 3R_1) \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -5 & 2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix}$$

24. $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 4 & 3 & -5 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow[\frac{1}{5}(R_1 - R_3) \to R_3]{4R_1 - R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 25 & 4 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & -\frac{1}{5} \end{array}\right]$$

$$\xrightarrow[R_2 - 25R_3 \to R_2]{R_1 - 5R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 5 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & -\frac{1}{5} \end{array}\right]$$

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -4 \\ 0 & 1 & 0 & -1 & -1 & 5 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & -\frac{1}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & -4 \\ -1 & -1 & 5 \\ \frac{1}{5} & 0 & -\frac{1}{5} \end{bmatrix}$$

25. $[A \mid I] = \left[\begin{array}{ccc|ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_1 - R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 3 & -4 & 2 & -3 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-(R_2 + 4R_3) \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

26. $[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_1 + R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 1 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array}\right]$$

$$\xrightarrow{R_1 + R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & 6 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

27. $ad - bc = (1)(2) - (0)(3) = 2$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

28. $ad - bc = (3)(6) - (4)(5) = -2$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix}$$

29. $ad - bc = (2)(5) - (-3)(-3) = 1$

$$A^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

30. $ad - bc = (3)(-2) - (-4)(2) = 2$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & \frac{3}{2} \end{bmatrix}$$

31. $ad - bc = (a)(-a) - (b)(-b) = -a^2 + b^2$

$$A^{-1} = \frac{1}{-a^2 + b^2}\begin{bmatrix} -a & b \\ -b & a \end{bmatrix} \text{ where } a^2 \neq b^2.$$

32. $ad - bc = (2)(-1) - (-1)(1) = -1$

$$A^{-1} = -1\begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

33. $$\begin{cases} 2x + 3y = -9 \\ x - 3y = 13 \end{cases} \Rightarrow \begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ 13 \end{bmatrix}$$

34. $$\begin{cases} 5x - 4y = 7 \\ 4x - 3y = 5 \end{cases} \Rightarrow \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

35. $$\begin{cases} 3x + 2y + z = 8 \\ 2x + y + 3z = 7 \\ x + 3y + 2z = 9 \end{cases} \Rightarrow \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 9 \end{bmatrix}$$

36. $$\begin{cases} x + 3y + z = 4 \\ x - 5y + 2z = 7 \\ 3x + y - 4z = -9 \end{cases} \Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 1 & -5 & 2 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -9 \end{bmatrix}$$

37. $$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} x - 2y = 0 \\ 2x + y = 5 \end{cases}$$

38. $$\begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix} \Rightarrow \begin{cases} 2x_1 + 3x_2 = 0 \\ 3x_1 - x_2 = 11 \end{cases}$$

39. $$\begin{bmatrix} 2 & 3 & 1 \\ 5 & 7 & -1 \\ 4 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} 2x_1 + 3x_2 + x_3 = -1 \\ 5x_1 + 7x_2 - x_3 = 5 \\ 4x_1 + 3x_2 = 5 \end{cases}$$

40. $$\begin{bmatrix} 3 & -2 & 3 \\ 5 & 0 & 4 \\ 2 & 7 & 0 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -8 \end{bmatrix} \Rightarrow \begin{cases} 3r - 2s + 3t = 4 \\ 5r + 4t = 3 \\ 2r + 7s = -8 \end{cases}$$

41. $$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

42. $$\begin{cases} x + 2y + 5z = 4 \\ 2x + 3y + 8z = 6 \\ -x + y + 2z = 3 \end{cases} \Rightarrow AX = B \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} \Rightarrow X = A^{-1}B \Rightarrow$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The solution is $\{(-1, 0, 1)\}$.

43. $$\begin{cases} x_1 + 2x_2 + 5x_3 = 1 \\ 2x_1 + 3x_2 + 8x_3 = 3 \\ -x_1 + x_2 + 2x_3 = -3 \end{cases} \Rightarrow AX = B \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \Rightarrow X = A^{-1}B \Rightarrow$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

The solution is $\{(2, -3, 1)\}$.

44. $$\begin{cases} x + 2y + 5z = -4 \\ 2x + 3y + 8z = -6 \\ -x + y + 2z = -\frac{5}{2} \end{cases} \Rightarrow AX = B \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 8 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ -\frac{5}{2} \end{bmatrix} \Rightarrow X = A^{-1}B \Rightarrow$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ 12 & -7 & -2 \\ -5 & 3 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ -6 \\ -\frac{5}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -1 \\ -\frac{1}{2} \end{bmatrix}$$

The solution is $\left\{\left(\frac{1}{2}, -1, -\frac{1}{2}\right)\right\}$.

45. a. $[A\,|\,I]=\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\1&2&3&0&1&0\\1&4&9&0&0&1\end{array}\right]$

$\xrightarrow[R_3-R_1\to R_3]{R_2-R_1\to R_2}\left[\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&2&-1&1&0\\0&3&8&-1&0&1\end{array}\right]$

$\xrightarrow[\frac{1}{2}(R_3-3R_2)\to R_3]{R_1-R_2\to R_1}\left[\begin{array}{ccc|ccc}1&0&-1&2&-1&0\\0&1&2&-1&1&0\\0&0&1&1&-\frac{3}{2}&\frac{1}{2}\end{array}\right]$

$\xrightarrow[R_2-2R_3\to R_2]{R_1+R_3\to R_1}\left[\begin{array}{ccc|ccc}1&0&0&3&-\frac{5}{2}&\frac{1}{2}\\0&1&0&-3&4&-1\\0&0&1&1&-\frac{3}{2}&\frac{1}{2}\end{array}\right]$

$$A^{-1}=\begin{bmatrix}3&-\frac{5}{2}&\frac{1}{2}\\-3&4&-1\\1&-\frac{3}{2}&\frac{1}{2}\end{bmatrix}$$

b. $\begin{cases}x+\ \ y+\ \ z=6\\x+2y+3z=14\\x+4y+9z=36\end{cases}\Rightarrow AX=B\Rightarrow$

$\begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\14\\36\end{bmatrix}\Rightarrow X=A^{-1}B\Rightarrow$

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3&-\frac{5}{2}&\frac{1}{2}\\-3&4&-1\\1&-\frac{3}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}6\\14\\36\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

The solution is $\{(1,2,3)\}$.

46.a. $[A\,|\,I]=\left[\begin{array}{ccc|ccc}2&4&-1&1&0&0\\3&1&2&0&1&0\\1&3&-3&0&0&1\end{array}\right]$

$\xrightarrow{R_1\leftrightarrow R_3}\left[\begin{array}{ccc|ccc}1&3&-3&0&0&1\\3&1&2&0&1&0\\2&4&-1&1&0&0\end{array}\right]$

$\xrightarrow[2R_1-R_3\to R_3]{3R_1-R_2\to R_2}\left[\begin{array}{ccc|ccc}1&3&-3&0&0&1\\0&8&-11&0&-1&3\\0&2&-5&-1&0&2\end{array}\right]$

$\xrightarrow{\frac{1}{9}(R_2-4R_3)\to R_3}\left[\begin{array}{ccc|ccc}1&3&-3&0&0&1\\0&8&-11&0&-1&3\\0&0&1&\frac{4}{9}&-\frac{1}{9}&-\frac{5}{9}\end{array}\right]$

$\xrightarrow[\frac{1}{8}(R_2+11R_3)\to R_2]{R_1+3R_3\to R_1}\left[\begin{array}{ccc|ccc}1&3&0&\frac{4}{3}&-\frac{1}{3}&-\frac{2}{3}\\0&1&0&\frac{11}{18}&-\frac{5}{18}&-\frac{7}{18}\\0&0&1&\frac{4}{9}&-\frac{1}{9}&-\frac{5}{9}\end{array}\right]$

$\xrightarrow{R_1-3R_2\to R_1}\left[\begin{array}{ccc|ccc}1&0&0&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&1&0&\frac{11}{18}&-\frac{5}{18}&-\frac{7}{18}\\0&0&1&\frac{4}{9}&-\frac{1}{9}&-\frac{5}{9}\end{array}\right]$

$$A^{-1}=\begin{bmatrix}-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\\frac{11}{18}&-\frac{5}{18}&-\frac{7}{18}\\\frac{4}{9}&-\frac{1}{9}&-\frac{5}{9}\end{bmatrix}$$

b. $\begin{cases}2x+4y-\ \ z=9\\3x+\ \ y+2z=7\\\ \ x+3y-3z=4\end{cases}\Rightarrow AX=B\Rightarrow$

$\begin{bmatrix}2&4&-1\\3&1&2\\1&3&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\7\\4\end{bmatrix}\Rightarrow X=A^{-1}B\Rightarrow$

$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\\frac{11}{18}&-\frac{5}{18}&-\frac{7}{18}\\\frac{4}{9}&-\frac{1}{9}&-\frac{5}{9}\end{bmatrix}\begin{bmatrix}9\\7\\4\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix}$

The solution is $\{(1,2,1)\}$.

9.3 Applying the Concepts

47. $\begin{cases} 3x+7y=17 \\ -5x+4y=13 \end{cases} \Rightarrow \begin{bmatrix} 3 & 7 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ -5 & 4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{5R_1+3R_2 \to R_2} \left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 0 & 47 & 5 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{47}R_2 \to R_2} \left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 0 & 1 & \frac{5}{47} & \frac{3}{47} \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}(R_1-7R_2) \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{47} & -\frac{7}{47} \\ 0 & 1 & \frac{5}{47} & \frac{3}{47} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{4}{47} & -\frac{7}{47} \\ \frac{5}{47} & \frac{3}{47} \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{4}{47} & -\frac{7}{47} \\ \frac{5}{47} & \frac{3}{47} \end{bmatrix}\begin{bmatrix} 11 \\ 13 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The solution is $\{(-1, 2)\}$.

48. $\begin{cases} x-7y=3 \\ 2x+3y=23 \end{cases} \Rightarrow \begin{bmatrix} 1 & -7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 23 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{cc|cc} 1 & -7 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{17}(R_2-2R_1) \to R_2} \left[\begin{array}{cc|cc} 1 & -7 & 1 & 0 \\ 0 & 1 & -\frac{2}{17} & \frac{1}{17} \end{array}\right]$$

$$\xrightarrow{(R_1+7R_2) \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{17} & \frac{7}{17} \\ 0 & 1 & -\frac{2}{17} & \frac{1}{17} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{3}{17} & \frac{7}{17} \\ -\frac{2}{17} & \frac{1}{17} \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{17} & \frac{7}{17} \\ -\frac{2}{17} & \frac{1}{17} \end{bmatrix}\begin{bmatrix} 3 \\ 23 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

The solution is $\{(10, 1)\}$.

49. $\begin{cases} x+\ \ y+2z=7 \\ x-\ \ y-3z=-6 \\ 2x+3y+\ \ z=4 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ 4 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & -3 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3-2R_1 \to R_3]{R_1-R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & -1 & 0 \\ 0 & 1 & -3 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{11}(R_2-2R_3) \to R_3]{2R_1-R_2 \to R_1} \left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 1 & 0 \\ 0 & 2 & 5 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{5}{11} & -\frac{1}{11} & -\frac{2}{11} \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}(R_2-5R_3) \to R_2]{\frac{1}{2}(R_1+R_3) \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{11} & \frac{5}{11} & -\frac{1}{11} \\ 0 & 1 & 0 & -\frac{7}{11} & -\frac{3}{11} & \frac{5}{11} \\ 0 & 0 & 1 & \frac{5}{11} & -\frac{1}{11} & -\frac{2}{11} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{8}{11} & \frac{5}{11} & -\frac{1}{11} \\ -\frac{7}{11} & -\frac{3}{11} & \frac{5}{11} \\ \frac{5}{11} & -\frac{1}{11} & -\frac{2}{11} \end{bmatrix}$$

$X = A^{-1}B \Rightarrow$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{11} & \frac{5}{11} & -\frac{1}{11} \\ -\frac{7}{11} & -\frac{3}{11} & \frac{5}{11} \\ \frac{5}{11} & -\frac{1}{11} & -\frac{2}{11} \end{bmatrix}\begin{bmatrix} 7 \\ -6 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

The solution is $\{(2, -1, 3\}\}$.

50. $\begin{cases} x+\ \ y+\ z=6 \\ 2x-3y+3z=5 \\ 3x-2y-\ z=-4 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 3 \\ 3 & -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -4 \end{bmatrix}$

$$[A\,|\,I] = \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 3 & -2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[3R_1-R_3\to R_3]{2R_1-R_2\to R_2} \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -1 & 2 & -1 & 0 \\ 0 & 5 & 4 & 3 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}(R_3-R_2)\to R_3} \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

$$\xrightarrow[\frac{1}{5}(R_2+R_3)\to R_2]{R_1-R_3\to R_1} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{1}{5} \\ 0 & 1 & 0 & \frac{11}{25} & -\frac{4}{25} & -\frac{1}{25} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

$$\xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{9}{5} & -\frac{1}{25} & \frac{6}{25} \\ 0 & 1 & 0 & \frac{11}{25} & -\frac{4}{25} & -\frac{1}{25} \\ 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{9}{5} & -\frac{1}{25} & \frac{6}{25} \\ \frac{11}{25} & -\frac{4}{25} & -\frac{1}{25} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

$X = A^{-1}B \Rightarrow$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{9}{5} & -\frac{1}{25} & \frac{6}{25} \\ \frac{11}{25} & -\frac{4}{25} & -\frac{1}{25} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The solution is $\{(1, 2, 3)\}$.

51. $\begin{cases} 2x+2y+3x=7 \\ 5x+3y+5z=3 \\ 3x+5y+\ z=-5 \end{cases} \Rightarrow \begin{bmatrix} 2 & 2 & 3 \\ 5 & 3 & 5 \\ 3 & 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix}$

$$[A\,|\,I] = \left[\begin{array}{rrr|rrr} 2 & 2 & 3 & 1 & 0 & 0 \\ 5 & 3 & 5 & 0 & 1 & 0 \\ 3 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[3R_1-2R_3\to R_3]{5R_1-2R_2\to R_2} \left[\begin{array}{rrr|rrr} 2 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 5 & 5 & -2 & 0 \\ 0 & -4 & 7 & 3 & 0 & -2 \end{array}\right]$$

$$\xrightarrow[\frac{1}{12}(R_2+R_3)\to R_3]{R_2-2R_1\to R_1} \left[\begin{array}{rrr|rrr} -4 & 0 & -1 & 3 & -2 & 0 \\ 0 & 4 & 5 & 5 & -2 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}(R_2-5R_3)\to R_2]{-\frac{1}{4}(R_1+R_3)\to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{11}{12} & \frac{13}{24} & \frac{1}{24} \\ 0 & 1 & 0 & \frac{5}{12} & -\frac{7}{24} & \frac{5}{24} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{11}{12} & \frac{13}{24} & \frac{1}{24} \\ \frac{5}{12} & -\frac{7}{24} & \frac{5}{24} \\ \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix}$$

$X = A^{-1}B \Rightarrow$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{11}{12} & \frac{13}{24} & \frac{1}{24} \\ \frac{5}{12} & -\frac{7}{24} & \frac{5}{24} \\ \frac{2}{3} & -\frac{1}{6} & -\frac{1}{6} \end{bmatrix}\begin{bmatrix} 7 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix}$$

The solution is $\{(-5, 1, 5)\}$.

52. $\begin{cases} 9x+7y+4z=12 \\ 6x+5y+4z=5 \\ 4x+3y+\ z=7 \end{cases} \Rightarrow \begin{bmatrix} 9 & 7 & 4 \\ 6 & 5 & 4 \\ 4 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 7 \end{bmatrix}$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 9 & 7 & 4 & 1 & 0 & 0 \\ 6 & 5 & 4 & 0 & 1 & 0 \\ 4 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[4R_1-9R_3\to R_3]{6R_1-9R_2\to R_2} \left[\begin{array}{ccc|ccc} 9 & 7 & 4 & 1 & 0 & 0 \\ 0 & -3 & -12 & 6 & -9 & 0 \\ 0 & 1 & 7 & 4 & 0 & -9 \end{array}\right]$$

$$\xrightarrow[\frac{1}{9}(R_2+3R_3)\to R_3]{R_1-R_3\to R_1} \left[\begin{array}{ccc|ccc} 9 & 6 & -3 & -3 & 0 & 9 \\ 0 & -3 & -12 & 6 & -9 & 0 \\ 0 & 0 & 1 & 2 & -1 & -3 \end{array}\right] \xrightarrow[-\frac{1}{3}(R_2+12R_3)\to R_2]{R_1+3R_3\to R_1} \left[\begin{array}{ccc|ccc} 9 & 6 & 0 & 3 & -3 & 0 \\ 0 & 1 & 0 & -10 & 7 & 12 \\ 0 & 0 & 1 & 2 & -1 & -3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}(R_1-6R_2)\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -5 & -8 \\ 0 & 1 & 0 & -10 & 7 & 12 \\ 0 & 0 & 1 & 2 & -1 & -3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 7 & -5 & -8 \\ -10 & 7 & 12 \\ 2 & -1 & -3 \end{bmatrix} \Rightarrow X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & -5 & -8 \\ -10 & 7 & 12 \\ 2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 12 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

The solution is $\{(3, -1, -2)\}$.

53. Let x = the amount invested in a treasury bill, y = the amount invested in bonds, and z = the amount invested in a mutual fund. Then we have

$$\begin{cases} x+y+z=90{,}000 \\ 0.03x+0.07y-0.11z=980 \\ 0.03x+0.07y+0.08z=5920 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0.03 & 0.07 & -0.11 \\ 0.03 & 0.07 & 0.08 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 90{,}000 \\ 980 \\ 5920 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ \frac{3}{100} & \frac{7}{100} & -\frac{11}{100} & 0 & 1 & 0 \\ \frac{3}{100} & \frac{7}{100} & \frac{8}{100} & 0 & 0 & 1 \end{array}\right] \xrightarrow[-\frac{3}{100}R_3+R_3\to]{-\frac{3}{100}R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{25} & -\frac{7}{50} & -\frac{3}{100} & 1 & 0 \\ 0 & \frac{1}{25} & \frac{1}{20} & -\frac{3}{100} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{100}{19}(R_2-R_3)\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{25} & -\frac{7}{50} & -\frac{3}{100} & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{100}{19} & \frac{100}{19} \end{array}\right] \xrightarrow{25R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{2} & -\frac{3}{4} & 25 & 0 \\ 0 & 0 & 1 & 0 & -\frac{100}{19} & \frac{100}{19} \end{array}\right]$$

$$\xrightarrow[R_2+\frac{7}{2}R_3\to R_2]{R_1-R_3\to R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & \frac{100}{19} & -\frac{100}{19} \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{125}{19} & \frac{350}{19} \\ 0 & 0 & 1 & 0 & -\frac{100}{19} & \frac{100}{19} \end{array}\right] \xrightarrow{R_1-R_2\to R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{7}{4} & -\frac{25}{19} & -\frac{450}{19} \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{125}{19} & \frac{350}{19} \\ 0 & 0 & 1 & 0 & -\frac{100}{19} & \frac{100}{19} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{7}{4} & -\frac{25}{19} & -\frac{450}{19} \\ -\frac{3}{4} & \frac{125}{19} & \frac{350}{19} \\ 0 & -\frac{100}{19} & \frac{100}{19} \end{bmatrix} \Rightarrow X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7}{4} & -\frac{25}{19} & -\frac{450}{19} \\ -\frac{3}{4} & \frac{125}{19} & \frac{350}{19} \\ 0 & -\frac{100}{19} & \frac{100}{19} \end{bmatrix}\begin{bmatrix} 90{,}000 \\ 980 \\ 5920 \end{bmatrix} = \begin{bmatrix} 16{,}000 \\ 48{,}000 \\ 26{,}000 \end{bmatrix}$$

Liz invested \$16,000 in a treasury bill, \$48,000 in bonds, and \$26,000 in a mutual fund.

54. Let x = the selling price of a chicken sandwich, y = the selling price of a fish sandwich, and x = the selling price of a ham sandwich. Then we have

$$\begin{cases} 8x+11y+6z=164 \\ 7x+8y+7z=141 \\ 4x+6y+3z=86 \end{cases} \Rightarrow \begin{bmatrix} 8 & 11 & 6 \\ 7 & 8 & 7 \\ 4 & 6 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 164 \\ 141 \\ 86 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 8 & 11 & 6 & 1 & 0 & 0 \\ 7 & 8 & 7 & 0 & 1 & 0 \\ 4 & 6 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-\frac{1}{2}R_1+R_3\to R_3]{-\frac{7}{8}R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 8 & 11 & 6 & 1 & 0 & 0 \\ 0 & -\frac{13}{8} & \frac{7}{4} & -\frac{7}{8} & 1 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 8 & 11 & 6 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 1 \\ 0 & -\frac{13}{8} & \frac{7}{4} & -\frac{7}{8} & 1 & 0 \end{array}\right] \xrightarrow{2R_2\to R_2} \left[\begin{array}{ccc|ccc} 8 & 11 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & -\frac{13}{8} & \frac{7}{4} & -\frac{7}{8} & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{13}{8}R_2+R_3\to R_3]{-11R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 8 & 0 & 6 & 12 & 0 & -22 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & \frac{7}{4} & -\frac{5}{2} & 1 & \frac{13}{4} \end{array}\right] \xrightarrow{\frac{4}{7}R_3\to R_3} \left[\begin{array}{ccc|ccc} 8 & 0 & 6 & 12 & 0 & -22 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{10}{7} & \frac{4}{7} & \frac{13}{7} \end{array}\right]$$

$$\xrightarrow{-6R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & \frac{144}{7} & -\frac{24}{7} & -\frac{232}{7} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{10}{7} & \frac{4}{7} & \frac{13}{7} \end{array}\right] \xrightarrow{\frac{1}{8}R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{18}{7} & -\frac{3}{7} & -\frac{29}{7} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{10}{7} & \frac{4}{7} & \frac{13}{7} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{18}{7} & -\frac{3}{7} & -\frac{29}{7} \\ -1 & 0 & 1 \\ -\frac{10}{7} & \frac{4}{7} & \frac{13}{7} \end{bmatrix} \Rightarrow X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{18}{7} & -\frac{3}{7} & -\frac{29}{7} \\ -1 & 0 & 1 \\ -\frac{10}{7} & \frac{4}{7} & \frac{13}{7} \end{bmatrix}\begin{bmatrix} 164 \\ 141 \\ 86 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 6 \end{bmatrix}$$

The chicken sandwich costs \$5, the fish sandwich costs \$8, and the ham sandwich costs \$6.

55. $$\begin{cases} V-E+R=2 \\ 2E=3V \\ 2R=E-1 \end{cases} \Rightarrow \begin{cases} V-E+R=2 \\ -3V+2E=0 \\ -E+2R=-1 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & 0 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -3 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{3R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 3 & 3 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-(R_3+R_2)\to R_3]{R_1+R_2\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -2 & -1 & 0 \\ 0 & 1 & -3 & -3 & -1 & 0 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2+3R_3\to R_2]{R_1+2R_3\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 1 & -2 \\ 0 & 1 & 0 & 6 & 2 & -3 \\ 0 & 0 & 1 & 3 & 1 & -1 \end{array}\right]$$

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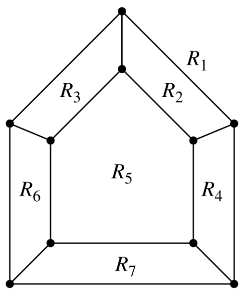
$$A^{-1}=\begin{bmatrix}4&1&-2\\6&2&-3\\3&1&-1\end{bmatrix}\Rightarrow X=A^{-1}B\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4&1&-2\\6&2&-3\\3&1&-1\end{bmatrix}\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}10\\15\\7\end{bmatrix}$$

There are 10 vertices, 15 edges, and 7 regions.



56. In order to answer (a) and (b), we must first determine the coefficients of the equation.

$$\begin{cases}\frac{1}{2}a+\ \ v_0+h_0=100\\2a+2v_0+h_0=92\\\frac{9}{2}a+3v_0+h_0=76\end{cases}\Rightarrow\begin{bmatrix}\frac{1}{2}&1&1\\2&2&1\\\frac{9}{2}&3&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}100\\92\\76\end{bmatrix}$$

$$[A\mid I]=\left[\begin{array}{ccc|ccc}\frac{1}{2}&1&1&1&0&0\\2&2&1&0&1&0\\\frac{9}{2}&3&1&0&0&1\end{array}\right]\xrightarrow[2R_3\to R_3]{2R_1\to R_1}\left[\begin{array}{ccc|ccc}1&2&2&2&0&0\\2&2&1&0&1&0\\9&6&2&0&0&2\end{array}\right]$$

$$\xrightarrow[-9R_1+R_3\to R_3]{-2R_1+R_2\to R_2}\left[\begin{array}{ccc|ccc}1&2&2&2&0&0\\0&-2&-3&-4&1&0\\0&-12&-16&-18&0&2\end{array}\right]\xrightarrow{\frac{1}{2}(-6R_2+R_3)\to R_3}\left[\begin{array}{ccc|ccc}1&2&2&2&0&0\\0&-2&-3&-4&1&0\\0&0&1&3&-3&1\end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}(3R_3+R_2)\to R_2]{-2R_3+R_1\to R_1}\left[\begin{array}{ccc|ccc}1&2&0&-4&6&-2\\0&1&0&-\frac{5}{2}&4&-\frac{3}{2}\\0&0&1&3&-3&1\end{array}\right]\xrightarrow{-2R_2+R_1\to R_1}\left[\begin{array}{ccc|ccc}1&0&0&1&-2&1\\0&1&0&-\frac{5}{2}&4&-\frac{3}{2}\\0&0&1&3&-3&1\end{array}\right]$$

$$A^{-1}=\begin{bmatrix}1&-2&1\\-\frac{5}{2}&4&-\frac{3}{2}\\3&-3&1\end{bmatrix}\Rightarrow X=A^{-1}B\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&-2&1\\-\frac{5}{2}&4&-\frac{3}{2}\\3&-3&1\end{bmatrix}\begin{bmatrix}100\\92\\76\end{bmatrix}=\begin{bmatrix}-8\\4\\100\end{bmatrix}$$

Thus, the equation is $h=f(t)=-4t^2+4t+100.$

a. $f(0)=-4(0)^2+4(0)+100=100$
The initial height is 100 feet.

b. $f(2.5)=-4(2.5)^2+4(2.5)+100=85$
After 2.5 seconds, the projectile is 85 feet high.

57. $X=\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0.1&0.4\\0.5&0.2\end{bmatrix}\right)^{-1}\begin{bmatrix}50\\30\end{bmatrix}$; $(I-A)^{-1}=\begin{bmatrix}\frac{9}{10}&-\frac{2}{5}\\-\frac{1}{2}&\frac{4}{5}\end{bmatrix}^{-1}$

$$\left[\begin{array}{cc|cc}\frac{9}{10}&-\frac{2}{5}&1&0\\-\frac{1}{2}&\frac{4}{5}&0&1\end{array}\right]\xrightarrow{\frac{1}{2}R_1+\frac{9}{10}R_2\to R_2}\left[\begin{array}{cc|cc}\frac{9}{10}&-\frac{2}{5}&1&0\\0&\frac{13}{25}&\frac{1}{2}&\frac{9}{10}\end{array}\right]\xrightarrow{\frac{13}{25}R_1+\frac{2}{5}R_2\to R_1}\left[\begin{array}{cc|cc}\frac{117}{250}&0&\frac{18}{25}&\frac{9}{25}\\0&\frac{13}{25}&\frac{1}{2}&\frac{9}{10}\end{array}\right]$$

$$\xrightarrow[\frac{25}{13}R_2\to R_2]{\frac{250}{117}R_1\to R_1}\left[\begin{array}{cc|cc}1&0&\frac{20}{13}&\frac{10}{13}\\0&1&\frac{25}{26}&\frac{45}{26}\end{array}\right]$$

$$(I-A)^{-1}=\begin{bmatrix}\frac{20}{13}&\frac{10}{13}\\\frac{25}{26}&\frac{45}{26}\end{bmatrix}\Rightarrow X=(I-A)^{-1}D=\begin{bmatrix}\frac{20}{13}&\frac{10}{13}\\\frac{25}{26}&\frac{45}{26}\end{bmatrix}\begin{bmatrix}50\\30\end{bmatrix}=\begin{bmatrix}100\\100\end{bmatrix}$$

58. $X=\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}0.2&0.5\\0.1&0.6\end{bmatrix}\right)^{-1}\begin{bmatrix}11\\18\end{bmatrix}$; $(I-A)^{-1}=\begin{bmatrix}\frac{4}{5}&-\frac{1}{2}\\-\frac{1}{10}&\frac{2}{5}\end{bmatrix}^{-1}$

$$\left[\begin{array}{cc|cc}\frac{4}{5}&-\frac{1}{2}&1&0\\-\frac{1}{10}&\frac{2}{5}&0&1\end{array}\right]\xrightarrow{\frac{1}{10}R_1+\frac{4}{5}R_2\to R_2}\left[\begin{array}{cc|cc}\frac{4}{5}&-\frac{1}{2}&1&0\\0&\frac{27}{100}&\frac{1}{10}&\frac{4}{5}\end{array}\right]\xrightarrow{\frac{27}{100}R_1+\frac{1}{2}R_2\to R_1}\left[\begin{array}{cc|cc}\frac{27}{125}&0&\frac{8}{25}&\frac{2}{5}\\0&\frac{27}{100}&\frac{1}{10}&\frac{4}{5}\end{array}\right]$$

$$\xrightarrow[\frac{100}{27}R_2\to R_2]{\frac{125}{27}R_1\to R_1}\left[\begin{array}{cc|cc}1&0&\frac{40}{27}&\frac{50}{27}\\0&1&\frac{10}{27}&\frac{80}{27}\end{array}\right]$$

$$(I-A)^{-1}=\begin{bmatrix}\frac{40}{27}&\frac{50}{27}\\\frac{10}{27}&\frac{80}{27}\end{bmatrix};\quad X=(I-A)^{-1}D=\begin{bmatrix}\frac{40}{27}&\frac{50}{27}\\\frac{10}{27}&\frac{80}{27}\end{bmatrix}\begin{bmatrix}11\\18\end{bmatrix}=\begin{bmatrix}\frac{1340}{27}\\\frac{1550}{27}\end{bmatrix}$$

59. $X=\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}0.2&0.2&0\\0.1&0.1&0.3\\0.1&0&0.2\end{bmatrix}\right)^{-1}\begin{bmatrix}400\\600\\800\end{bmatrix}$

$$(I-A)^{-1}=\begin{bmatrix}\frac{4}{5}&-\frac{1}{5}&0\\-\frac{1}{10}&\frac{9}{10}&-\frac{3}{10}\\-\frac{1}{10}&0&\frac{4}{5}\end{bmatrix}^{-1}\Rightarrow\left[\begin{array}{ccc|ccc}\frac{4}{5}&-\frac{1}{5}&0&1&0&0\\-\frac{1}{10}&\frac{9}{10}&-\frac{3}{10}&0&1&0\\-\frac{1}{10}&0&\frac{4}{5}&0&0&1\end{array}\right]\xrightarrow[\frac{1}{10}R_1+\frac{4}{5}R_3\to R_3]{\frac{1}{10}R_1+\frac{4}{5}R_2\to R_2}\left[\begin{array}{ccc|ccc}\frac{4}{5}&-\frac{1}{5}&0&1&0&0\\0&\frac{7}{10}&-\frac{6}{25}&\frac{1}{10}&\frac{4}{5}&0\\0&-\frac{1}{50}&\frac{16}{25}&\frac{1}{10}&0&\frac{4}{5}\end{array}\right]$$

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$$\xrightarrow{\frac{625}{277}\left(\frac{1}{50}R_2+\frac{7}{10}R_3\right)\to R_3}\left[\begin{array}{ccc|ccc} \frac{4}{5} & -\frac{1}{5} & 0 & 1 & 0 & 0 \\ 0 & \frac{7}{10} & -\frac{6}{25} & \frac{1}{10} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & \frac{45}{277} & \frac{10}{277} & \frac{350}{277} \end{array}\right]$$

$$\xrightarrow[\frac{10}{7}\left(\frac{6}{25}R_3+R_2\to R_2\right)]{5R_1\to R_1}\left[\begin{array}{ccc|ccc} 4 & -1 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & \frac{55}{277} & \frac{320}{277} & \frac{120}{277} \\ 0 & 0 & 1 & \frac{45}{277} & \frac{10}{277} & \frac{350}{277} \end{array}\right]\xrightarrow{\frac{1}{4}(R_1+R_2)\to R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{360}{277} & \frac{80}{277} & \frac{30}{277} \\ 0 & 1 & 0 & \frac{55}{277} & \frac{320}{277} & \frac{120}{277} \\ 0 & 0 & 1 & \frac{45}{277} & \frac{10}{277} & \frac{350}{277} \end{array}\right]\Rightarrow$$

$$(I-A)^{-1}=\begin{bmatrix} \frac{360}{277} & \frac{80}{277} & \frac{30}{277} \\ \frac{55}{277} & \frac{320}{277} & \frac{120}{277} \\ \frac{45}{277} & \frac{10}{277} & \frac{350}{277} \end{bmatrix};\quad X=(I-A)^{-1}D=\begin{bmatrix} \frac{360}{277} & \frac{80}{277} & \frac{30}{277} \\ \frac{55}{277} & \frac{320}{277} & \frac{120}{277} \\ \frac{45}{277} & \frac{10}{277} & \frac{350}{277} \end{bmatrix}\begin{bmatrix} 400 \\ 600 \\ 800 \end{bmatrix}=\begin{bmatrix} \frac{216,000}{277} \\ \frac{310,000}{277} \\ \frac{304,000}{277} \end{bmatrix}$$

60. $$X=\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.1 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}\right)^{-1}\begin{bmatrix} 500 \\ 300 \\ 200 \end{bmatrix}$$

$$(I-A)^{-1}=\begin{bmatrix} \frac{1}{2} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{7}{10} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{1}{10} & \frac{7}{10} \end{bmatrix}^{-1}\Rightarrow\left[\begin{array}{ccc|ccc} \frac{1}{2} & -\frac{2}{5} & -\frac{1}{5} & 1 & 0 & 0 \\ -\frac{1}{5} & \frac{7}{10} & -\frac{1}{10} & 0 & 1 & 0 \\ -\frac{1}{10} & -\frac{1}{10} & \frac{7}{10} & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{100}{27}\left(\frac{9}{100}R_3+R_2\right)\to R_2]{R_1+2R_3\to R_1}\left[\begin{array}{ccc|ccc} 5 & -4 & 0 & \frac{100}{9} & \frac{10}{9} & \frac{10}{3} \\ 0 & 1 & 0 & \frac{25}{27} & \frac{55}{27} & \frac{5}{9} \\ 0 & 0 & 1 & \frac{5}{9} & \frac{5}{9} & \frac{5}{3} \end{array}\right]\xrightarrow{\frac{1}{5}(R_1+4R_2)\to R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{80}{27} & \frac{50}{27} & \frac{10}{9} \\ 0 & 1 & 0 & \frac{25}{27} & \frac{55}{27} & \frac{5}{9} \\ 0 & 0 & 1 & \frac{5}{9} & \frac{5}{9} & \frac{5}{3} \end{array}\right]\Rightarrow$$

$$\xrightarrow[\frac{1}{10}R_1+\frac{1}{2}R_3\to R_3]{\frac{1}{5}R_1+\frac{1}{2}R_2\to R_2}\left[\begin{array}{ccc|ccc} \frac{1}{2} & -\frac{2}{5} & -\frac{1}{5} & 1 & 0 & 0 \\ 0 & \frac{27}{100} & -\frac{9}{100} & \frac{1}{5} & \frac{1}{2} & 0 \\ 0 & -\frac{9}{100} & \frac{33}{100} & \frac{1}{10} & 0 & \frac{1}{2} \end{array}\right]\xrightarrow[\frac{10}{9}(3R_3+R_2)\to R_3]{10R_1\to R_1}\left[\begin{array}{ccc|ccc} 5 & -4 & -2 & 10 & 0 & 0 \\ 0 & \frac{27}{100} & -\frac{9}{100} & \frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{9} & \frac{5}{9} & \frac{5}{3} \end{array}\right]$$

$$(I-A)^{-1}=\begin{bmatrix} \frac{80}{27} & \frac{50}{27} & \frac{10}{9} \\ \frac{25}{27} & \frac{55}{27} & \frac{5}{9} \\ \frac{5}{9} & \frac{5}{9} & \frac{5}{3} \end{bmatrix};\ X=(I-A)^{-1}D=\begin{bmatrix} \frac{80}{27} & \frac{50}{27} & \frac{10}{9} \\ \frac{25}{27} & \frac{55}{27} & \frac{5}{9} \\ \frac{5}{9} & \frac{5}{9} & \frac{5}{3} \end{bmatrix}\begin{bmatrix} 500 \\ 300 \\ 200 \end{bmatrix}=\begin{bmatrix} \frac{61,000}{27} \\ \frac{32,000}{27} \\ \frac{7000}{9} \end{bmatrix}$$

Answers will vary in exercises 61−64.

61.a. Associate each letter in the phrase with the number representing its position in the alphabet, and partition the numbers into groups of two, forming a 2 × 9 matrix:

$$M = \begin{bmatrix} 3 & 14 & 15 & 6 & 14 & 3 & 12 & 13 & 15 \\ 1 & 14 & 20 & 9 & 4 & 15 & 15 & 2 & 0 \end{bmatrix} \text{ Let } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ Then,}$$

$$AM = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 14 & 15 & 6 & 14 & 3 & 12 & 13 & 15 \\ 1 & 14 & 20 & 9 & 4 & 15 & 15 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 42 & 55 & 24 & 22 & 33 & 42 & 17 & 15 \\ 7 & 42 & 50 & 21 & 32 & 21 & 39 & 28 & 30 \end{bmatrix}$$

The cryptogram is 5 7 42 42 55 50 24 21 22 32 33 21 42 39 17 28 15 30.

b. $$A^{-1} = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}(2R_1 - R_2)\to R_2} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{array}\right] = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$M = A^{-1}(AM) = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 5 & 42 & 55 & 24 & 22 & 33 & 42 & 17 & 15 \\ 7 & 42 & 50 & 21 & 32 & 21 & 39 & 28 & 30 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 14 & 15 & 6 & 14 & 3 & 12 & 13 & 15 \\ 1 & 14 & 20 & 9 & 4 & 15 & 15 & 2 & 0 \end{bmatrix}$$

62.a. Associate each letter in the phrase with the number representing its position in the alphabet, and partition the numbers into groups of two, forming a 2 × 9 matrix:

$$M = \begin{bmatrix} 6 & 21 & 4 & 5 & 4 & 6 & 5 & 18 & 18 & 19 & 14 & 20 & 15 & 11 \\ 15 & 14 & 8 & 1 & 15 & 20 & 18 & 15 & 9 & 20 & 5 & 23 & 18 & 0 \end{bmatrix} \text{ Let } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{ Then,}$$

$$AM = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 6 & 21 & 4 & 5 & 4 & 6 & 5 & 18 & 18 & 19 & 14 & 20 & 15 & 11 \\ 15 & 14 & 8 & 1 & 15 & 20 & 18 & 15 & 9 & 20 & 5 & 23 & 18 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 36 & 49 & 20 & 7 & 34 & 46 & 41 & 48 & 36 & 59 & 24 & 66 & 51 & 11 \\ 27 & 56 & 16 & 11 & 23 & 32 & 28 & 51 & 45 & 58 & 33 & 63 & 48 & 22 \end{bmatrix}$$

The cryptogram is 36 27 49 56 20 16 7 11 34 23 46 32 41 28 48 51 36 45 59 58 24 33 66 63 51 48 11 22.

b. From exercise 61b, $A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

$$M = A^{-1}(AM) = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 36 & 49 & 20 & 7 & 34 & 46 & 41 & 48 & 36 & 59 & 24 & 66 & 51 & 11 \\ 27 & 56 & 16 & 11 & 23 & 32 & 28 & 51 & 45 & 58 & 33 & 63 & 48 & 22 \end{bmatrix}$$
$$= \begin{bmatrix} 36 & 49 & 20 & 7 & 34 & 46 & 41 & 48 & 36 & 59 & 24 & 66 & 51 & 11 \\ 27 & 56 & 16 & 11 & 23 & 32 & 28 & 51 & 45 & 58 & 33 & 63 & 48 & 22 \end{bmatrix}$$

63.a. Associate each letter in the phrase with the number representing its position in the alphabet, and partition the numbers into groups of three, forming a 3 × 6 matrix:

$$M = \begin{bmatrix} 3 & 14 & 6 & 4 & 12 & 2 \\ 1 & 15 & 9 & 3 & 15 & 15 \\ 14 & 20 & 14 & 15 & 13 & 0 \end{bmatrix} \text{ Let } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \text{ Then,}$$

$$AM = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 14 & 6 & 4 & 12 & 2 \\ 1 & 15 & 9 & 3 & 15 & 15 \\ 14 & 20 & 14 & 15 & 13 & 0 \end{bmatrix} = \begin{bmatrix} 21 & 63 & 35 & 26 & 52 & 19 \\ 19 & 64 & 38 & 25 & 55 & 32 \\ 32 & 69 & 43 & 37 & 53 & 17 \end{bmatrix}$$

The cryptogram is 21 39 32 63 64 69 35 38 43 26 25 37 52 55 53 19 32 17.

b. $$A^{-1} = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1-2R_3\to R_3]{R_1-2R_2\to R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & 1 & -2 & 0 \\ 0 & -1 & -3 & 1 & 0 & -2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{8}(R_2-3R_3)\to R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right] \xrightarrow[-\frac{1}{3}(R_3+R_2)\to R_2]{R_1-R_3\to R_1} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & \frac{5}{4} & \frac{1}{4} & -\frac{3}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}(R_1-R_2)\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{array}\right] = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$M = A^{-1}(AM) = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 21 & 63 & 35 & 26 & 52 & 19 \\ 19 & 64 & 38 & 25 & 55 & 32 \\ 32 & 69 & 43 & 37 & 53 & 17 \end{bmatrix} = \begin{bmatrix} 3 & 14 & 6 & 4 & 12 & 2 \\ 1 & 15 & 9 & 3 & 15 & 15 \\ 14 & 20 & 14 & 15 & 13 & 0 \end{bmatrix}$$

64.a. Associate each letter in the phrase with the number representing its position in the alphabet, and partition the numbers into groups of three, forming a 3 × 9 matrix:

$$M = \begin{bmatrix} 6 & 14 & 5 & 15 & 5 & 15 & 19 & 5 & 15 \\ 15 & 4 & 1 & 6 & 18 & 18 & 20 & 20 & 18 \\ 21 & 8 & 4 & 20 & 18 & 9 & 14 & 23 & 11 \end{bmatrix} \text{ Let } A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \text{ Then,}$$

$$AM = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 & 14 & 5 & 15 & 5 & 15 & 19 & 5 & 15 \\ 15 & 4 & 1 & 6 & 18 & 18 & 20 & 20 & 18 \\ 21 & 8 & 4 & 20 & 18 & 9 & 14 & 23 & 11 \end{bmatrix} = \begin{bmatrix} 48 & 40 & 15 & 56 & 46 & 57 & 72 & 53 & 59 \\ 57 & 30 & 11 & 47 & 59 & 60 & 73 & 68 & 62 \\ 63 & 34 & 14 & 61 & 59 & 51 & 67 & 71 & 55 \end{bmatrix}$$

The cryptogram is 48 57 63 40 30 34 15 11 14 56 47 61 46 59 59 57 60 51 72 73 67 53 68 71 59 62 55.

b. From exercise 59b, $A^{-1}=\begin{bmatrix}\frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4}\end{bmatrix}$

$$M=A^{-1}(AM)=\begin{bmatrix}\frac{3}{4} & -\frac{1}{4} & -\frac{1}{4}\\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4}\\ -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4}\end{bmatrix}\begin{bmatrix}48 & 40 & 15 & 56 & 46 & 57 & 72 & 53 & 59\\ 57 & 30 & 11 & 47 & 59 & 60 & 73 & 68 & 62\\ 63 & 34 & 14 & 61 & 59 & 51 & 67 & 71 & 55\end{bmatrix}$$
$$=\begin{bmatrix}6 & 14 & 5 & 15 & 5 & 15 & 19 & 5 & 15\\ 15 & 4 & 1 & 6 & 18 & 18 & 20 & 20 & 18\\ 21 & 8 & 4 & 20 & 18 & 9 & 14 & 23 & 11\end{bmatrix}$$

9.3 Beyond the Basics

65.a. Yes, by the definition of *inverse* if $AB = I$, then $BA = I$. So $B = A^{-1}$ and $A = B^{-1}$.

b. $\left(A^{-1}\right)^{-1} = A$

66. $A^{-1}=\begin{bmatrix}2 & 1\\ 3 & -4\end{bmatrix}\Rightarrow A=\frac{1}{2(-4)-3(1)}\begin{bmatrix}-4 & -1\\ -3 & 2\end{bmatrix}$
$=\begin{bmatrix}\frac{4}{11} & \frac{1}{11}\\ \frac{3}{11} & -\frac{2}{11}\end{bmatrix}$

67. $I = A^2B = A(AB)$, so AB is the inverse of A.

68. $I = AB = AIB = A(AB)B = A^2(B^2)$, so B^2 is the inverse of A^2.

69. $ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$
$B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I \Rightarrow AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

70. $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\Rightarrow A^{-1}=-\frac{1}{2}\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}=\begin{bmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}$

$B=\begin{bmatrix}-1 & 0\\ 3 & -2\end{bmatrix}\Rightarrow$

$B^{-1}=\frac{1}{2}\begin{bmatrix}-2 & 0\\ -3 & -1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ -\frac{3}{2} & -\frac{1}{2}\end{bmatrix}$

$AB=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}-1 & 0\\ 3 & -2\end{bmatrix}=\begin{bmatrix}5 & -4\\ 9 & -8\end{bmatrix}$

$(AB)^{-1}=-\frac{1}{4}\begin{bmatrix}-8 & 4\\ -9 & 5\end{bmatrix}=\begin{bmatrix}2 & -1\\ \frac{9}{4} & -\frac{5}{4}\end{bmatrix}$

$B^{-1}A^{-1}=\begin{bmatrix}-1 & 0\\ -\frac{3}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}2 & -1\\ \frac{9}{4} & -\frac{5}{4}\end{bmatrix}$

71. If A is invertible, then there exists a matrix D such that $AD = I$ and $DA = I$. Then $AB = AC \Leftrightarrow DAB = DAC \Leftrightarrow IB = IC \Leftrightarrow B = C$.

72. Not necessarily. For example, let $A=\begin{bmatrix}9 & 6\\ 6 & 4\end{bmatrix}$, $B=\begin{bmatrix}2 & 4\\ 3 & 6\end{bmatrix}$, and $C=\begin{bmatrix}4 & 8\\ 0 & 0\end{bmatrix}$.

A^{-1} does not exist $(ad-bc=0)$.

$AB=AC=\begin{bmatrix}36 & 72\\ 24 & 48\end{bmatrix}$.

73. If A is invertible, then
$I = A^{-1}A = A^{-1}(AB) = \left(A^{-1}A\right)B = IB = B.$

74. Let $A=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 3 & 4\end{bmatrix}$. A^{-1} does not exist $(ad-bc=0)$. $AB=\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}$.

However, $B \neq I$.

75.a. $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^2 - 6\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 24 \\ 12 & 17 \end{bmatrix} - \begin{bmatrix} 18 & 24 \\ 12 & 18 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

b. $A^2 - 6A + I = 0 \Rightarrow I = 6A - A^2 \Rightarrow AA^{-1} = 6A - A^2 \Rightarrow A^{-1} = 6I - A$

c. $A^{-1} = 6I - A = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

76.a. $A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b. $A^3 - 6A^2 + 9A - 4I = 0 \Rightarrow A^3 - 6A^2 + 9A - 4AA^{-1} = 0 \Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0 \Rightarrow$
$4A^{-1} = A^2 - 6A + 9I \Rightarrow A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$

c. $A^{-1} = \frac{1}{4}\left(A^2 - 6A + 9I\right) = \frac{1}{4}\left(\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}^2 - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$

$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

77. $\begin{cases} x - y - z - w = -4 \\ x + y + z - w = 2 \\ 2x + y + z - w = 3 \\ x - y + z - w = -2 \end{cases} \Rightarrow \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 3 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -4 \\ 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

Using a graphing calculator, we have

$\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 2 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$

Solution set: $\{(1, 2, 1, 2)\}$

78. $\begin{cases} x+y+z-w=1 \\ x-y-z+w=1 \\ x-y+z+3w=1 \\ x+y+z+w=3 \end{cases} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$

Using a graphing calculator, we have

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution set: $\{(1, 2, -1, 1)\}$

79.a. $ABB^{-1} = AI = A \Rightarrow A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 6 & 8 \end{bmatrix}$

b. $B^{-1}BA = IA = A \Rightarrow A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ -3 & 17 \end{bmatrix}$

c. $B^{-1}BAB^{-1} = IAB^{-1} = AB^{-1}$; $\qquad AB^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix} \Rightarrow \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix} \Rightarrow \begin{cases} a+3b=3 \\ 2a+4b=5 \\ c+3d=7 \\ 2c+4d=11 \end{cases} \Rightarrow$$

$$a = \frac{3}{2}, b = \frac{1}{2}, c = \frac{5}{2}, d = \frac{3}{2} \Rightarrow A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & \frac{3}{2} \end{bmatrix}$$

d. $B^{-1}ABB^{-1} = B^{-1}AI = B^{-1}A \Rightarrow B^{-1}A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 10 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 10 \end{bmatrix} \Rightarrow \begin{cases} a+2c=4 \\ b+2d=6 \\ 3a+4c=7 \\ 3b+4d=10 \end{cases} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 7 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 6 \\ 7 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ 7 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ \frac{5}{2} \\ 4 \end{bmatrix} \Rightarrow$$

$$A = \begin{bmatrix} -1 & -2 \\ \frac{5}{2} & 4 \end{bmatrix}$$

80.a. Yes. $ad - bd = x^2 + 2x + 2$. $x^2 + 2x + 2 = 0$ has no real zeros, so the matrix is invertible.

b. No. $ad - bc = x^2 - 6x + 8.$
$x^2 - 6x + 8 = 0 \Rightarrow x = 2$ or $x = 4,$ so the matrix is not invertible for those values of x.

9.3 Critical Thinking/Discussion/Writing

81.a. True. $A^2B = AAB = ABA = BAA = BA^2$

b. False. For example, if $A = I$ and $B = -I$, then $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is not invertible.

82. Let $A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$(I + A)^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

83. True. A matrix has an inverse if and only if it is square.

9.3 Maintaining Skills

84. 1

85. 1

86. −1

87. −1

88. $\begin{bmatrix} -1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \left[(-1)2 + 4(3)\right] = [10]$

89. $\begin{bmatrix} -1 \\ -3 \end{bmatrix}\begin{bmatrix} 5 & 0 \end{bmatrix} = \begin{bmatrix} -1(5) & -1(0) \\ -3(5) & -3(0) \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ -15 & 0 \end{bmatrix}$

90. $3\begin{bmatrix} 7 & 1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 21 & 3 \\ 15 & -6 \end{bmatrix}$

91. $(-1)\begin{bmatrix} -6 & 7 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -7 \\ -4 & 1 \end{bmatrix}$

In exercises 92–95, be sure to check the solution in the original equations.

92. $\begin{cases} 2x - 3y = 16 & (1) \\ x - y = 7 & (2) \end{cases}$

Solve equation (2) for x in terms of y.
$x = y + 7$
Substitute the expression for x in equation (1) and solve for y.
$2(y + 7) - 3y = 16 \Rightarrow 2y + 14 - 3y = 16 \Rightarrow$
$-y = 2 \Rightarrow y = -2$

Substitute $y = -2$ in equation (2) and solve for x.
$x - (-2) = 7 \Rightarrow x = 5$
Solution set: {(5, −2)}

93. $\begin{cases} 16x - 9y = -5 & (1) \\ 10x + 18y = -11 & (2) \end{cases}$

Multiply equation (1) by 2, then add the resulting equation and equation (2).

$$\begin{array}{r} 32x - 18y = -10 \\ \underline{10x + 18y = -11} \\ 42x = -21 \Rightarrow x = -\dfrac{1}{2} \end{array}$$

Substitute $x = -\frac{1}{2}$ in equation (2) and solve for y.

$10\left(-\frac{1}{2}\right) + 18y = -11 \Rightarrow -5 + 18y = -11 \Rightarrow$
$18y = -6 \Rightarrow y = -\frac{1}{3}$

Solution set: $\left\{\left(-\frac{1}{2}, -\frac{1}{3}\right)\right\}$

94. $\begin{cases} x - y + 5z = -6 & (1) \\ 3x + 3y - z = 10 & (2) \\ x + 3y + 2z = 5 & (3) \end{cases}$

Write the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$$

Now use Gauss-Jordan elimination to solve the system.

$$\left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 3 & 3 & -1 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right]$$

$$\xrightarrow[-R_3 + R_1 \to R_3]{-3R_1 + R_2 \to R_2} \left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & -4 & 3 & -11 \end{array}\right]$$

$$\xrightarrow{\frac{2}{3}R_2 + R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 0 & -\frac{23}{3} & \frac{23}{3} \end{array}\right]$$

$$\xrightarrow{-\frac{3}{23}R_3 \to R_3} \left[\begin{array}{rrr|r} 1 & -1 & 5 & -6 \\ 0 & 6 & -16 & 28 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

(*continued on next page*)

(*continued*)

$$\xrightarrow[\frac{1}{6}(16R_3+R_2)\to R_3]{R_1-5R_3\to R_1}\left[\begin{array}{ccc|c}1 & -1 & 0 & -1\\0 & 1 & 0 & 2\\0 & 0 & 1 & -1\end{array}\right]$$

$$\xrightarrow{R_1+R_1\to R_1}\left[\begin{array}{ccc|c}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & -1\end{array}\right]$$

Solution set: $\{(1, 2, -1)\}$

95. $\begin{cases}2x - \ \ y + 2z = 3 & (1)\\2x + 2y - \ \ z = 0 & (2)\\-x + 2y + 2z = -12 & (3)\end{cases}$

Write the augmented matrix.

$$\left[\begin{array}{ccc|c}2 & -1 & 2 & 3\\2 & 2 & -1 & 0\\-1 & 2 & 2 & -12\end{array}\right]$$

Now use Gauss-Jordan elimination to solve the system.

$$\left[\begin{array}{ccc|c}2 & -1 & 2 & 3\\2 & 2 & -1 & 0\\-1 & 2 & 2 & -12\end{array}\right]$$

$$\xrightarrow[R_1+2R_3\to R_3]{R_2-R_1\to R_2}\left[\begin{array}{ccc|c}2 & -1 & 2 & 3\\0 & 3 & -3 & -3\\0 & 3 & 6 & -21\end{array}\right]$$

$$\xrightarrow{\frac{1}{9}(R_3-R_2)\to R_3}\left[\begin{array}{ccc|c}2 & -1 & 2 & 3\\0 & 3 & -3 & -3\\0 & 0 & 1 & -2\end{array}\right]$$

$$\xrightarrow[\frac{1}{3}(R_2+3R_3)\to R_2]{R_1-2R_3\to R_1}\left[\begin{array}{ccc|c}2 & -1 & 0 & 7\\0 & 1 & 0 & -3\\0 & 0 & 1 & -2\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}(R_1+R_2)\to R_1}\left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\0 & 1 & 0 & -3\\0 & 0 & 1 & -2\end{array}\right]$$

Solution set: $\{(2, -3, -2)\}$

9.4 Determinants and Cramer's Rule

9.4 Practice Problems

1. a. $\begin{vmatrix}5 & 1\\3 & -7\end{vmatrix} = 5(-7) - 1(3) = -38$

b. $\begin{vmatrix}2 & -9\\-4 & 18\end{vmatrix} = 2(18) - (-9)(-4) = 0$

2. a. $M_{11} = \begin{bmatrix}3 & -1 & 2\\4 & 5 & 6\\7 & 1 & 2\end{bmatrix} = \begin{vmatrix}5 & 6\\1 & 2\end{vmatrix}$

$= (5)(2) - 6(1) = 4$

$M_{23} = \begin{bmatrix}3 & -1 & 2\\4 & 5 & 6\\7 & 1 & 2\end{bmatrix} = \begin{vmatrix}3 & -1\\7 & 1\end{vmatrix}$

$= 3(1) - (-1)(7) = 10$

$M_{32} = \begin{bmatrix}3 & -1 & 2\\4 & 5 & 6\\7 & 1 & 2\end{bmatrix} = \begin{vmatrix}3 & 2\\4 & 6\end{vmatrix}$

$= 3(6) - 2(4) = 10$

b. $A_{11} = (-1)^{1+1}M_{11} = 4$

$A_{23} = (-1)^{2+3}M_{23} = -10$

$A_{32} = (-1)^{3+2}M_{32} = -10$

3. Expand by the third row:

$\begin{vmatrix}2 & -3 & 7\\-2 & -1 & 9\\0 & 2 & -9\end{vmatrix}$

$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$

$= 0 + 2(-1)^{3+2}\begin{vmatrix}2 & 7\\-2 & 9\end{vmatrix} - 9(-1)^{3+3}\begin{vmatrix}2 & -3\\-2 & -1\end{vmatrix}$

$= 0 - 2(32) - 9(-8) = 8$

4. $D = \begin{vmatrix}2 & 3\\5 & 9\end{vmatrix} = 3,\ D_x = \begin{vmatrix}7 & 3\\4 & 9\end{vmatrix} = 51$

$D_y = \begin{vmatrix}2 & 7\\5 & 4\end{vmatrix} = -27$

$x = \dfrac{D_x}{D} = \dfrac{51}{3} = 17;\ y = \dfrac{D_y}{D} = \dfrac{-27}{3} = -9$

The solution is $\{(17, -9)\}$.

5. $D = \begin{vmatrix}3 & 2 & 1\\4 & 3 & 1\\5 & 1 & 1\end{vmatrix} = -3,\ D_x = \begin{vmatrix}4 & 2 & 1\\5 & 3 & 1\\9 & 1 & 1\end{vmatrix} = -6$

$D_y = \begin{vmatrix}3 & 4 & 1\\4 & 5 & 1\\5 & 9 & 1\end{vmatrix} = 3,\ D_z = \begin{vmatrix}3 & 2 & 4\\4 & 3 & 5\\5 & 1 & 9\end{vmatrix} = 0$

$x = \dfrac{D_x}{D} = \dfrac{-6}{-3} = 2;\ y = \dfrac{D_y}{D} = \dfrac{3}{-3} = -1$

$z = \dfrac{D_z}{D} = \dfrac{0}{-3} = 0$

Solution set: $\{(2, -1, 0)\}$

9.4 Basic Concepts and Skills

1. The determinant of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\underline{ad - bc}$.

2. The minor of an element is the determinant you get by deleting the <u>row</u> and <u>column</u> containing that element.

3. To expand an $n \times n$ determinant, you multiply each element of some row (or column) by its <u>cofactor</u> and add the result.

4. A system of n linear equations in n variables has a unique solution provided the determinant of the <u>coefficient matrix</u> is not zero.

5. False.

6. False. Each element is multiplied by its cofactor.

7. $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2(5) - 4(3) = -2$

8. $\begin{vmatrix} 3 & -5 \\ 1 & 4 \end{vmatrix} = 3(4) - (-5)(1) = 17$

9. $\begin{vmatrix} 4 & -2 \\ 3 & -3 \end{vmatrix} = 4(-3) - (-2)(3) = -6$

10. $\begin{vmatrix} -2 & \frac{1}{2} \\ 3 & 1 \end{vmatrix} = (-2)(1) - \left(\frac{1}{2}\right)(3) = -\frac{7}{2}$

11. $\begin{vmatrix} -1 & -3 \\ -4 & -5 \end{vmatrix} = (-1)(-5) - (-3)(-4) = -7$

12. $\begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{vmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = 0$

13. $\begin{vmatrix} \frac{3}{8} & \frac{1}{2} \\ -\frac{1}{9} & 5 \end{vmatrix} = \left(\frac{3}{8}\right)(5) - \left(\frac{1}{2}\right)\left(-\frac{1}{9}\right) = \frac{139}{72}$

14. $\begin{vmatrix} -\frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & -\frac{1}{3} \end{vmatrix} = \left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) - \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$

15. $\begin{vmatrix} \sqrt{a} & \sqrt{b} \\ \sqrt{b} & \sqrt{a} \end{vmatrix} = a - b$

16. $\begin{vmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{vmatrix} = \left(\sqrt{3}\right)^2 - 1(-1) = 4$

17. a. $M_{21} = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{vmatrix} -3 & 4 \\ 1 & 2 \end{vmatrix}$
$= (-3)(2) - 4(1) = -10$

b. $M_{23} = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} = 2(1) - 0 = 2$

c. $M_{32} = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} = 2(2) - 4(1) = 0$

18. a. $A_{21} = (-1)^{2+1} M_{21} = 10$

b. $A_{23} = (-1)^{2+3} M_{23} = -2$

c. $A_{32} = (-1)^{3+2} M_{32} = 0$

19. a. $M_{11} = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}$
$= (-1)(2) - 2(1) = -4$

b. $M_{22} = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} = 4$

c. $M_{31} = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} = \begin{vmatrix} -3 & 4 \\ -1 & 2 \end{vmatrix}$
$= (-3)(2) - 4(-1) = -2$

20. a. $A_{11} = (-1)^{1+1} M_{11} = -4$

b. $A_{22} = (-1)^{2+2} M_{22} = 4$

c. $A_{31} = (-1)^{3+1} M_{31} = -2$

21. Expand by the third row:

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & 0 & 0 \end{vmatrix} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$
$$= -1(-1)^{3+1}\begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} + 0 + 0 = -2$$

22. Expand by the second column:

$$\begin{vmatrix} 2 & 0 & \frac{1}{2} \\ 1 & 0 & 2 \\ 4 & 0 & -5 \end{vmatrix} = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = 0$$

23. Expand by the third row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{vmatrix} = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$
$$= 0 + 0 + 4(-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 12$$

24. Expand by the first column:

$$\begin{vmatrix} 2 & 3 & 4 \\ 0 & -4 & 6 \\ 0 & 0 & -5 \end{vmatrix} = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$$
$$= 2(-1)^{1+1}\begin{vmatrix} -4 & 6 \\ 0 & -5 \end{vmatrix} + 0 + 0 = 40$$

25. Expand by the first row:

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 5 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 1(-1)^{1+1}\begin{vmatrix} 0 & 0 \\ 4 & 5 \end{vmatrix} + 0 + 0 = 0$$

26. Expand by the first row:

$$\begin{vmatrix} -1 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & -3 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= -1(-1)^{1+1}\begin{vmatrix} 2 & 0 \\ 5 & -3 \end{vmatrix} + 0 + 0 = 6$$

27. Expand by the first row:

$$\begin{vmatrix} 1 & 6 & 0 \\ 2 & 5 & 3 \\ 3 & 4 & 0 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 1(-1)^{1+1}\begin{vmatrix} 5 & 3 \\ 4 & 0 \end{vmatrix} + 6(-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} + 0$$
$$= -12 - 6(-9) = 42$$

28. Expand by the first row:

$$\begin{vmatrix} 1 & 0 & 2 \\ 0 & 5 & 7 \\ 3 & 1 & 0 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 1(-1)^{1+1}\begin{vmatrix} 5 & 7 \\ 1 & 0 \end{vmatrix} + 0 + 2(-1)^{1+3}\begin{vmatrix} 0 & 5 \\ 3 & 1 \end{vmatrix}$$
$$= -7 + 2(-15) = -37$$

29. Expand by the first row:

$$\begin{vmatrix} 3 & 4 & 1 \\ 1 & 4 & 3 \\ 4 & 3 & 1 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 3(-1)^{1+1}\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} + 4(-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix}$$
$$+ 1(-1)^{1+3}\begin{vmatrix} 1 & 4 \\ 4 & 3 \end{vmatrix}$$
$$= 3(-5) - 4(-11) + 1(-13) = 16$$

30. Expand by the second row:

$$\begin{vmatrix} 3 & 1 & -2 \\ 4 & 0 & -4 \\ 2 & -1 & -3 \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$= 4(-1)^{2+1}\begin{vmatrix} 1 & -2 \\ -1 & -3 \end{vmatrix} + 0 - 4(-1)^{2+3}\begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix}$$
$$= -4(-5) + 0 + 4(-5) = 0$$

31. Expand by the first row:

$$\begin{vmatrix} 0 & 1 & 6 \\ 1 & 0 & 4 \\ 8 & 3 & 1 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 0 + 1(-1)^{1+2}\begin{vmatrix} 1 & 4 \\ 8 & 1 \end{vmatrix} + 6(-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 8 & 3 \end{vmatrix}$$
$$= -1(-31) + 6(3) = 49$$

32. Expand by the first row:

$$\begin{vmatrix} 0 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 0 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
$$= 0 + 2(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix}$$
$$= -2(-3) + 3(2) = 12$$

33. Expand by the first row:

$$\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= a(-1)^{1+1}\begin{vmatrix} a & b \\ 0 & a \end{vmatrix} + b(-1)^{1+2}\begin{vmatrix} 0 & b \\ b & a \end{vmatrix} + 0$$

$$= a(a^2) - b(-b^2) = a^3 + b^3$$

34. Expand by the first row:

$$\begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= 0 + z(-1)^{1+2}\begin{vmatrix} z & x \\ y & 0 \end{vmatrix} + y(-1)^{1+3}\begin{vmatrix} z & 0 \\ y & x \end{vmatrix}$$

$$= -z(-xy) + y(xz) = 2xyz$$

35. Expand by the first row:

$$\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= a(-1)^{1+1}\begin{vmatrix} a & b \\ c & a \end{vmatrix} + b(-1)^{1+2}\begin{vmatrix} c & b \\ b & a \end{vmatrix} + c(-1)^{1+3}\begin{vmatrix} c & a \\ b & c \end{vmatrix}$$

$$= a(a^2 - bc) - b(ac - b^2) + c(c^2 - ab)$$

$$= a^3 + b^3 + c^3 - 3abc$$

36. Expand by the first row:

$$\begin{vmatrix} u & v & w \\ w & v & u \\ u & v & w \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= u(-1)^{1+1}\begin{vmatrix} v & u \\ v & w \end{vmatrix} + v(-1)^{1+2}\begin{vmatrix} w & u \\ u & w \end{vmatrix} + w(-1)^{1+3}\begin{vmatrix} w & v \\ u & v \end{vmatrix}$$

$$= u(vw - vu) - v(w^2 - u^2) + w(vw - uv)$$

$$= 0$$

37. $D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2, D_x = \begin{vmatrix} 8 & 1 \\ -2 & -1 \end{vmatrix} = -6$

$D_y = \begin{vmatrix} 1 & 8 \\ 1 & -2 \end{vmatrix} = -10$

$x = \frac{-6}{-2} = 3, y = \frac{-10}{-2} = 5$

The solution is $\{(3, 5)\}$.

38. $D = \begin{vmatrix} 4 & 3 \\ 2 & -5 \end{vmatrix} = -26, D_x = \begin{vmatrix} -1 & 3 \\ 9 & -5 \end{vmatrix} = -22$

$D_y = \begin{vmatrix} 4 & -1 \\ 2 & 9 \end{vmatrix} = 38$

$x = \frac{-22}{-26} = \frac{11}{13}, y = \frac{38}{-26} = -\frac{19}{13}$

The solution is $\left\{\left(\frac{11}{13}, -\frac{19}{13}\right)\right\}$.

39. $D = \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = -1, D_x = \begin{vmatrix} 11 & 3 \\ 4 & 1 \end{vmatrix} = -1$

$D_y = \begin{vmatrix} 5 & 11 \\ 2 & 4 \end{vmatrix} = -2$

$x = \frac{-1}{-1} = 1, y = \frac{-2}{-1} = 2$

The solution is $\{(1, 2)\}$.

40. $D = \begin{vmatrix} 2 & -7 \\ 5 & 6 \end{vmatrix} = 47, D_x = \begin{vmatrix} 13 & -7 \\ 9 & 6 \end{vmatrix} = 141$

$D_y = \begin{vmatrix} 2 & 13 \\ 5 & 9 \end{vmatrix} = -47$

$x = \frac{141}{47} = 3, y = \frac{-47}{47} = -1$

The solution is $\{(3, -1)\}$.

41. $D = \begin{vmatrix} 2 & 9 \\ 3 & -2 \end{vmatrix} = -31, D_x = \begin{vmatrix} 4 & 9 \\ 6 & -2 \end{vmatrix} = -62$

$D_y = \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0$

$x = \frac{-62}{-31} = 2, y = \frac{0}{-31} = 0$

The solution is $\{(2, 0)\}$.

42. $D = \begin{vmatrix} 5 & 3 \\ 2 & -5 \end{vmatrix} = -31, D_x = \begin{vmatrix} 1 & 3 \\ -12 & -5 \end{vmatrix} = 31$

$D_y = \begin{vmatrix} 5 & 1 \\ 2 & -12 \end{vmatrix} = -62$

$x = \frac{31}{-31} = -1, y = \frac{-62}{-31} = 2$

The solution is $\{(-1, 2)\}$.

43. $D = \begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix} = 0 \Rightarrow$ there is not a unique solution. $2x - 3y = 4 \Rightarrow x = \frac{3}{2}y + 2.$

The solution is $\left\{\left(\frac{3}{2}y + 2, y\right)\right\}$.

44. $D = \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 0 \Rightarrow$ there is not a unique solution. $3x + y = 2 \Rightarrow y = -3x + 2$. The solution is $\{(x, -3x + 2)\}$.

45. Let $u = \frac{1}{x}$ and $v = \frac{1}{y}$. Then $\begin{cases} \frac{2}{x} + \frac{3}{y} = 2 \\ \frac{5}{x} + \frac{8}{y} = \frac{31}{6} \end{cases} \Rightarrow$

$\begin{cases} 2u + 3v = 2 \\ 5u + 8v = \frac{31}{6} \end{cases} \Rightarrow D = \begin{vmatrix} 2 & 3 \\ 5 & 8 \end{vmatrix} = 1,$

$D_u = \begin{vmatrix} 2 & 3 \\ \frac{31}{6} & 8 \end{vmatrix} = \frac{1}{2}, D_v = \begin{vmatrix} 2 & 2 \\ 5 & \frac{31}{6} \end{vmatrix} = \frac{1}{3} \Rightarrow$

$u = \frac{1}{2}, v = \frac{1}{3} \Rightarrow x = 2, y = 3$

The solution is $\{(2, 3)\}$.

46. Let $u = \frac{1}{x}$ and $v = \frac{1}{y}$. Then

$\begin{cases} \frac{3}{x} - \frac{6}{y} = 2 \\ \frac{4}{x} + \frac{7}{y} = -3 \end{cases} \Rightarrow \begin{cases} 3u - 6v = 2 \\ 4u + 7v = -3 \end{cases} \Rightarrow$

$D = \begin{vmatrix} 3 & -6 \\ 4 & 7 \end{vmatrix} = 45,\ D_u = \begin{vmatrix} 2 & -6 \\ -3 & 7 \end{vmatrix} = -4,$

$D_v = \begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix} = -17 \Rightarrow u = -\frac{4}{45}, v = -\frac{17}{45} \Rightarrow$

$x = -\frac{45}{4}, y = -\frac{45}{17}$

The solution is $\left\{\left(-\frac{45}{4}, -\frac{45}{17}\right)\right\}$.

47. $D = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 11, D_x = \begin{vmatrix} -1 & -2 & 1 \\ 4 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 11$

$D_y = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 4 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 11, D_z = \begin{vmatrix} 1 & -2 & -1 \\ 3 & 1 & 4 \\ 0 & 1 & 1 \end{vmatrix} = 0$

$x = \frac{11}{11} = 1, y = \frac{11}{11} = 1, z = \frac{0}{11} = 0$

The solution is $\{(1, 1, 0)\}$.

48. $D = \begin{vmatrix} 1 & 3 & 0 \\ 0 & 1 & 3 \\ 4 & 0 & 1 \end{vmatrix} = 37, D_x = \begin{vmatrix} 4 & 3 & 0 \\ 7 & 1 & 3 \\ 6 & 0 & 1 \end{vmatrix} = 37$

$D_y = \begin{vmatrix} 1 & 4 & 0 \\ 0 & 7 & 3 \\ 4 & 6 & 1 \end{vmatrix} = 37, D_z = \begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 7 \\ 4 & 0 & 6 \end{vmatrix} = 74$

$x = \frac{37}{37} = 1, y = \frac{37}{37} = 1, z = \frac{74}{37} = 2$

The solution is $\{(1, 1, 2)\}$.

49. $D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 0 & 2 & 1 \end{vmatrix} = -5, D_x = \begin{vmatrix} -3 & 1 & -1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -5$

$D_y = \begin{vmatrix} 1 & -3 & -1 \\ 2 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 5, D_z = \begin{vmatrix} 1 & 1 & -3 \\ 2 & 3 & 2 \\ 0 & 2 & 1 \end{vmatrix} = -15$

$x = \frac{-5}{-5} = 1, y = \frac{5}{-5} = -1, z = \frac{-15}{-5} = 3$

The solution is $\{(1, -1, 3)\}$.

50. $D = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 5 & -9 & 0 \end{vmatrix} = -46, D_x = \begin{vmatrix} 10 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -9 & 0 \end{vmatrix} = -92$

$D_y = \begin{vmatrix} 3 & 10 & 1 \\ 1 & 0 & -1 \\ 5 & 1 & 0 \end{vmatrix} = -46, D_z = \begin{vmatrix} 3 & 1 & 10 \\ 1 & 1 & 0 \\ 5 & -9 & 1 \end{vmatrix} = -138$

$x = \frac{-92}{-46} = 2, y = \frac{-46}{-46} = 1, z = \frac{-138}{-46} = 3$

The solution is $\{(2, 1, 3)\}$.

51. $D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & 5 \\ 1 & 2 & -4 \end{vmatrix} = 22, D_x = \begin{vmatrix} 3 & 1 & 1 \\ 4 & -3 & 5 \\ -1 & 2 & -4 \end{vmatrix} = 22$

$D_y = \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 5 \\ 1 & -1 & -4 \end{vmatrix} = 22, D_z = \begin{vmatrix} 1 & 1 & 3 \\ 2 & -3 & 4 \\ 1 & 2 & -1 \end{vmatrix} = 22$

$x = \frac{22}{22} = 1, y = \frac{22}{22} = 1, z = \frac{22}{22} = 1$

The solution is $\{(1, 1, 1)\}$.

52. $D = \begin{vmatrix} 1 & 2 & -1 \\ 3 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -10, D_x = \begin{vmatrix} 3 & 2 & -1 \\ 8 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -30$

$D_y = \begin{vmatrix} 1 & 3 & -1 \\ 3 & 8 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 10, D_z = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & 8 \\ 1 & 1 & 0 \end{vmatrix} = 20$

$x = \frac{-30}{-10} = 3, y = \frac{10}{-10} = -1, z = \frac{20}{-10} = -2$

The solution is $\{(3, -1, -2)\}\}$.

53. $D = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 5 & -2 \\ 1 & 2 & -3 \end{vmatrix} = -38$

$D_x = \begin{vmatrix} 11 & -3 & 5 \\ 7 & 5 & -2 \\ -4 & 2 & -3 \end{vmatrix} = -38$

$D_y = \begin{vmatrix} 2 & 11 & 5 \\ 3 & 7 & -2 \\ 1 & -4 & -3 \end{vmatrix} = -76$

$D_z = \begin{vmatrix} 2 & -3 & 11 \\ 3 & 5 & 7 \\ 1 & 2 & -4 \end{vmatrix} = -114$

$x = \frac{-38}{-38} = 1, y = \frac{-76}{-38} = 2, z = \frac{-114}{-38} = 3$

The solution is $\{(1, 2, 3)\}$.

54. $D = \begin{vmatrix} 1 & -3 & 0 \\ 2 & -1 & -4 \\ 0 & 1 & 2 \end{vmatrix} = 14, D_x = \begin{vmatrix} 1 & -3 & 0 \\ 2 & -1 & -4 \\ 4 & 1 & 2 \end{vmatrix} = 62$

$D_y = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 2 & -4 \\ 0 & 4 & 2 \end{vmatrix} = 16, D_z = \begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix} = 20$

$x = \frac{62}{14} = \frac{31}{7}, y = \frac{16}{14} = \frac{8}{7}, z = \frac{20}{14} = \frac{10}{7}$

The solution is $\left\{\left(\frac{31}{7}, \frac{8}{7}, \frac{10}{7}\right)\right\}$.

55. $D = \begin{vmatrix} 5 & 2 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{vmatrix} = -7, D_x = \begin{vmatrix} 12 & 2 & 1 \\ 13 & 1 & 3 \\ 19 & 2 & 4 \end{vmatrix} = -7$

$D_y = \begin{vmatrix} 5 & 12 & 1 \\ 2 & 13 & 3 \\ 3 & 19 & 4 \end{vmatrix} = -14, D_z = \begin{vmatrix} 5 & 2 & 12 \\ 2 & 1 & 13 \\ 3 & 2 & 19 \end{vmatrix} = -21$

$x = \frac{-7}{-7} = 1, y = \frac{-14}{-7} = 2, z = \frac{-21}{-7} = 3$

The solution is $\{(1, 2, 3)\}$.

56. $D = \begin{vmatrix} 2 & 1 & 1 \\ 3 & -1 & -1 \\ 1 & 2 & -3 \end{vmatrix} = 25, D_x = \begin{vmatrix} 7 & 1 & 1 \\ -2 & -1 & -1 \\ -4 & 2 & -3 \end{vmatrix} = 25$

$D_y = \begin{vmatrix} 2 & 7 & 1 \\ 3 & -2 & -1 \\ 1 & -4 & -3 \end{vmatrix} = 50, D_z = \begin{vmatrix} 2 & 1 & 7 \\ 3 & -1 & -2 \\ 1 & 2 & -4 \end{vmatrix} = 75$

$x = \frac{25}{25} = 1, y = \frac{50}{25} = 2, z = \frac{75}{25} = 3$

The solution is $\{(1, 2, 3)\}$.

9.4 Applying the Concepts

57. $A = |D| = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ -3 & 4 & 1 \\ 4 & 6 & 1 \end{vmatrix} = 11$

58. $A = |D| = \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 4 & 2 & 1 \\ 5 & 4 & 1 \end{vmatrix} = \frac{1}{2}$

59. $A = |D| = \frac{1}{2}\begin{vmatrix} -2 & 1 & 1 \\ -3 & -5 & 1 \\ 2 & 4 & 1 \end{vmatrix} = \frac{21}{2} = 10.5$

60. $A = |D| = \frac{1}{2}\begin{vmatrix} -1 & 2 & 1 \\ 2 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4$

61. $\begin{vmatrix} 0 & 3 & 1 \\ -1 & 1 & 1 \\ 2 & 7 & 1 \end{vmatrix} = 0 \Rightarrow$ the points are collinear.

62. $\begin{vmatrix} 2 & 0 & 1 \\ 1 & \frac{1}{2} & 1 \\ 4 & -1 & 1 \end{vmatrix} = 0 \Rightarrow$ the points are collinear.

63. $\begin{vmatrix} 0 & -4 & 1 \\ 3 & -2 & 1 \\ 1 & -4 & 1 \end{vmatrix} = -2 \Rightarrow$ the points are not collinear.

64. $\begin{vmatrix} 0 & -\frac{1}{4} & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} = -\frac{1}{4} \Rightarrow$ the points are not collinear.

65. $\begin{vmatrix} x & y & 1 \\ -1 & -1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = 0 \Rightarrow x(-1)^{1+1}\begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} + y(-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} = 0 \Rightarrow -4x + 2y - 2 = 0 \Rightarrow y = 2x + 1$

66. $\begin{vmatrix} x & y & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow x(-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} + y(-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 0 & 4 \\ 1 & 1 \end{vmatrix} = 0 \Rightarrow 3x + y - 4 = 0 \Rightarrow y = -3x + 4$

67. $\begin{vmatrix} x & y & 1 \\ 0 & \frac{1}{3} & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow x(-1)^{1+1}\begin{vmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{vmatrix} + y(-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 0 & \frac{1}{3} \\ 1 & 1 \end{vmatrix} = 0 \Rightarrow -\frac{2}{3}x + y - \frac{1}{3} = 0 \Rightarrow y = \frac{2}{3}x + \frac{1}{3}$

68. $\begin{vmatrix} x & y & 1 \\ 1 & -\frac{1}{3} & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0 \Rightarrow x(-1)^{1+1}\begin{vmatrix} -\frac{1}{3} & 1 \\ 0 & 1 \end{vmatrix} + y(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 1 & -\frac{1}{3} \\ 2 & 1 \end{vmatrix} = 0 \Rightarrow -\frac{1}{3}x + y + \frac{2}{3} = 0 \Rightarrow y = \frac{1}{3}x - \frac{2}{3}$

9.4 Beyond the Basics

69. a. $\begin{vmatrix} 0 & 0 \\ 2 & 5 \end{vmatrix} = 0(5) - 2(0) = 0$

b. $\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 1(0) - 3(0) = 0$

c. Expand by the second row: $\begin{vmatrix} 1 & -2 & -3 \\ 0 & 0 & 0 \\ 4 & 4 & -7 \end{vmatrix} = 0(-1)^{2+1}\begin{vmatrix} -2 & -3 \\ 4 & -7 \end{vmatrix} + 0(-1)^{2+2}\begin{vmatrix} 1 & -3 \\ 4 & -7 \end{vmatrix} + 0(-1)^{2+3}\begin{vmatrix} 1 & -2 \\ 4 & 4 \end{vmatrix} = 0$

d. Expand by the third column: $\begin{vmatrix} 4 & 5 & 0 \\ 6 & -7 & 0 \\ 8 & 15 & 0 \end{vmatrix} = 0(-1)^{1+3}\begin{vmatrix} 6 & -7 \\ 8 & 15 \end{vmatrix} + 0(-1)^{2+3}\begin{vmatrix} 4 & 5 \\ 8 & 15 \end{vmatrix} + 0(-1)^{3+3}\begin{vmatrix} 4 & 5 \\ 6 & -7 \end{vmatrix} = 0$

70. a. $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2, |B| = \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 2 \Rightarrow |B| = -|A|$

b. $|A| = \begin{vmatrix} -5 & 3 \\ 2 & -4 \end{vmatrix} = 14, |B| = \begin{vmatrix} 3 & -5 \\ -4 & 2 \end{vmatrix} = -14 \Rightarrow |B| = -|A|$

c. $|A| = \begin{vmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 3 & -1 & 4 \end{vmatrix} = 9, |B| = \begin{vmatrix} 1 & 3 & 5 \\ 3 & -1 & 4 \\ 0 & 1 & 2 \end{vmatrix} = -9 \Rightarrow |B| = -|A|$

71. a. Expand by the second row: $\begin{vmatrix} 2 & 3 & 5 \\ -1 & 4 & 8 \\ 2 & 3 & 5 \end{vmatrix} = -1(-1)^{2+1}\begin{vmatrix} 3 & 5 \\ 3 & 5 \end{vmatrix} + 4(-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} + 8(-1)^{2+3}\begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} = 0$

b. Expand by the second column: $\begin{vmatrix} 3 & 4 & 3 \\ 1 & 2 & 1 \\ -1 & 6 & -1 \end{vmatrix} = 4(-1)^{1+2}\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} + 2(-1)^{2+2}\begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix} + 6(-1)^{3+2}\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix} = 0$

72. a. $|B| = \begin{vmatrix} -1 & 2 & 3 \\ 1\cdot 5 & 4\cdot 5 & 8\cdot 5 \\ 2 & 3 & 6 \end{vmatrix} = 25;\quad 5|A| = \begin{vmatrix} -1 & 2 & 3 \\ 1 & 4 & 8 \\ 2 & 3 & 6 \end{vmatrix} = 5(5) = 25 \Rightarrow |B| = 5|A|$

b. $|B| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 6 \\ 0 & 2 & 9 \end{vmatrix} = 45;\quad 3|A| = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 5 & 2 \\ 0 & 2 & 3 \end{vmatrix} = 3(15) = 45 \Rightarrow |B| = 3|A|$

73. $\begin{vmatrix} 2 & -3 & 4 \\ -4 & 7 & -8 \\ 5 & -1 & 3 \end{vmatrix} = -14,\quad \begin{vmatrix} 2 & -3 & 4 \\ 0 & 1 & 0 \\ 5 & -1 & 3 \end{vmatrix} = -14$

74.(i) $\begin{vmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 2 & 1 \\ 1 & 2 & 5 & 3 \\ 2 & 4 & 3 & 7 \end{vmatrix} \xrightarrow{-R_1+R_3\to R_3} \begin{vmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 4 & 3 & 7 \end{vmatrix}$

(ii) $\begin{vmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 4 & 3 & 7 \end{vmatrix} = 0(-1)^{3+1}\begin{vmatrix} 2 & 3 & 3 \\ 5 & 2 & 1 \\ 4 & 3 & 7 \end{vmatrix} + 0(-1)^{3+2}\begin{vmatrix} 1 & 3 & 3 \\ 4 & 2 & 1 \\ 2 & 3 & 7 \end{vmatrix} + 2(-1)^{3+3}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 4 & 7 \end{vmatrix} + 0(-1)^{3+4}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 2 & 4 & 3 \end{vmatrix}$

(iii) $\begin{vmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 2 & 4 & 3 & 7 \end{vmatrix} = 0 + 0 + 2(-1)^{3+3}\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 4 & 7 \end{vmatrix} + 0 = 2(-3) = -6$

75. $\begin{vmatrix} 2 & 0 & -3 & 4 \\ 0 & 1 & 0 & 5 \\ 5 & 0 & -9 & 8 \\ 1 & 2 & 0 & 7 \end{vmatrix} \xrightarrow[R_3-5R_4\to R_3]{R_1-2R_4\to R_1} \begin{vmatrix} 0 & -4 & -3 & -10 \\ 0 & 1 & 0 & 5 \\ 0 & -10 & -9 & -27 \\ 1 & 2 & 0 & 7 \end{vmatrix}$

Expand by column 1:

$$\begin{vmatrix} 0 & -4 & -3 & -10 \\ 0 & 1 & 0 & 5 \\ 0 & -10 & -9 & -27 \\ 1 & 2 & 0 & 7 \end{vmatrix} = 0 + 0 + 0 + 1(-1)^{4+1}\begin{vmatrix} -4 & -3 & -10 \\ 1 & 0 & 5 \\ -10 & -9 & -27 \end{vmatrix} = 21$$

76. $\begin{vmatrix} -3 & 1 & 2 & 3 \\ 0 & -2 & 4 & 7 \\ -9 & 3 & 5 & 2 \\ 2 & -4 & 3 & -12 \end{vmatrix} \xrightarrow{3R_1 - R_3 \to R_3} \begin{vmatrix} -3 & 1 & 2 & 3 \\ 0 & -2 & 4 & 7 \\ 0 & 0 & 1 & 7 \\ 2 & -4 & 3 & -12 \end{vmatrix} \xrightarrow[4R_1 + R_4 \to R_4]{2R_1 + R_2 \to R_2} \begin{vmatrix} -3 & 1 & 2 & 3 \\ -6 & 0 & 8 & 13 \\ 0 & 0 & 1 & 7 \\ -10 & 0 & 11 & 0 \end{vmatrix}$

Expand by column 2: $\begin{vmatrix} -3 & 1 & 2 & 3 \\ -6 & 0 & 8 & 13 \\ 0 & 0 & 1 & 7 \\ -10 & 0 & 11 & 0 \end{vmatrix} = 1(-1)^{3+3} \begin{vmatrix} -6 & 8 & 13 \\ 0 & 1 & 7 \\ -10 & 11 & 0 \end{vmatrix} = 32$

77. $\begin{vmatrix} 5 & 7 & 1 & 2 \\ 6 & 8 & 9 & 3 \\ 24 & 22 & 6 & 10 \\ 21 & 17 & 7 & 10 \end{vmatrix} \xrightarrow[\substack{-6R_1 + R_3 \to R_3 \\ -7R_1 + R_4 \to R_4}]{-9R_1 + R_2 \to R_2} \begin{vmatrix} 5 & 7 & 1 & 2 \\ -39 & -55 & 0 & -15 \\ -6 & -20 & 0 & -2 \\ -14 & -32 & 0 & -4 \end{vmatrix}$

Expand by column 3: $\begin{vmatrix} 5 & 7 & 1 & 2 \\ -39 & -55 & 0 & -15 \\ -6 & -20 & 0 & -2 \\ -14 & -32 & 0 & -4 \end{vmatrix} = 1(-1)^{1+3} \begin{vmatrix} -39 & -55 & -15 \\ -6 & -20 & -2 \\ -14 & -32 & -4 \end{vmatrix} = 476$

78. $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & -3 & 1 & 0 \\ -1 & -3 & 4 & 5 \\ 1 & 3 & 7 & 4 \end{vmatrix} \xrightarrow[\substack{R_1 + R_3 \to R_3 \\ R_4 - R_1 \to R_4}]{R_2 - 2R_1 \to R_2} \begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & -7 & -5 & 0 \\ 0 & -1 & 7 & 5 \\ 0 & 1 & 4 & 4 \end{vmatrix}$

Expand by column 1: $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 0 & -7 & -5 & 0 \\ 0 & -1 & 7 & 5 \\ 0 & 1 & 4 & 4 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} -7 & -5 & 0 \\ -1 & 7 & 5 \\ 1 & 4 & 4 \end{vmatrix} = -101$

79. $\begin{vmatrix} 3 & 2 \\ 6 & x \end{vmatrix} = 0 \Rightarrow 3x - 12 = 0 \Rightarrow x = 4$

80. $\begin{vmatrix} 3 & 2 \\ x & 12 \end{vmatrix} = 0 \Rightarrow 36 - 2x = 0 \Rightarrow x = 18$

81. $\begin{vmatrix} x & -2 \\ 1 & x-1 \end{vmatrix} = 0 \Rightarrow x^2 - x + 2 = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{1-8}}{2} = \dfrac{1 \pm i\sqrt{7}}{2}$

82. $\begin{vmatrix} 2x+7 & 4 \\ 1 & x \end{vmatrix} = 0 \Rightarrow 2x^2 + 7x - 4 = 0 \Rightarrow (x+4)(2x-1) = 0 \Rightarrow x = -4 \text{ or } x = \dfrac{1}{2}$

83. Expand by the second row:

$\begin{vmatrix} 1 & -3 & 1 \\ 4 & 7 & x \\ 0 & 2 & 2 \end{vmatrix} = 0 \Rightarrow 4(-1)^{2+1} \begin{vmatrix} -3 & 1 \\ 2 & 2 \end{vmatrix} + 7(-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + x(-1)^{2+3} \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 0 \Rightarrow$

$-4(-8) + 7(2) - 2x = 46 - 2x = 0 \Rightarrow x = 23$

84. Expand by the first row:

$$\begin{vmatrix} x & x+1 & x+2 \\ 2 & 3 & -1 \\ 3 & -2 & 4 \end{vmatrix} = 0 \Rightarrow x(-1)^{1+1}\begin{vmatrix} 3 & -1 \\ -2 & 4 \end{vmatrix} + (x+1)(-1)^{1+2}\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} + (x+2)(-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = 0 \Rightarrow$$

$$10x - 11(x+1) - 13(x+2) = -14x - 37 = 0 \Rightarrow x = -\frac{37}{14}$$

85. Expand by the second row:

$$\begin{vmatrix} x & 0 & 1 \\ 0 & x & 0 \\ 1 & 0 & x \end{vmatrix} = 0 \Rightarrow 0 + x(-1)^{2+2}\begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} + 0 \Rightarrow x(x^2-1) = 0 \Rightarrow x(x-1)(x+1) = 0 \Rightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1$$

86. Expand by the second row:

$$\begin{vmatrix} x & 2 & 3 \\ x & x & 1 \\ 2 & 0 & 1 \end{vmatrix} = -8 \Rightarrow x(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} + x(-1)^{2+2}\begin{vmatrix} x & 3 \\ 2 & 1 \end{vmatrix} + 1(-1)^{2+3}\begin{vmatrix} x & 2 \\ 2 & 0 \end{vmatrix} = -8 \Rightarrow$$

$$-2x + x(x-6) + 4 = x^2 - 8x + 4 = -8 \Rightarrow x^2 - 8x + 12 = 0 \Rightarrow (x-2)(x-6) = 0 \Rightarrow x = 2 \text{ or } x = 6$$

87. $$\frac{1}{2}\begin{vmatrix} -4 & 2 & 1 \\ 0 & k & 1 \\ -2 & k & 1 \end{vmatrix} = 28 \Rightarrow 0 + k(-1)^{2+2}\begin{vmatrix} -4 & 1 \\ -2 & 1 \end{vmatrix} + 1(-1)^{2+3}\begin{vmatrix} -4 & 2 \\ -2 & k \end{vmatrix} = 56 \Rightarrow -2k - (-4k+4) = 2k - 4 = 56 \Rightarrow k = 30$$

88. $$\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ -2 & k & 1 \end{vmatrix} = 3 \Rightarrow -2(-1)^{3+1}\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} + k(-1)^{3+2}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 1(-1)^{3+3}\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 6 \Rightarrow$$

$$-2(1) - k + 1 = -1 - k = 6 \Rightarrow k = -7$$

9.4 Critical Thinking/Discussion/Writing

89. Let $u = \dfrac{1}{x-2}$ and $v = \dfrac{1}{y+1}$. Then $\begin{cases} \dfrac{1}{x-2} + \dfrac{3}{y+1} = 13 \\ \dfrac{4}{x-2} - \dfrac{5}{y+1} = 1 \end{cases} \Rightarrow \begin{cases} u + 3v = 13 \\ 4u - 5v = 1 \end{cases}$.

$$D = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -17, \quad D_u = \begin{vmatrix} 13 & 3 \\ 1 & -5 \end{vmatrix} = -68, \quad D_v = \begin{vmatrix} 1 & 13 \\ 4 & 1 \end{vmatrix} = -51$$

$$u = \frac{-68}{-14}, \quad v = \frac{-51}{-17} = 3$$

$$4 = \frac{1}{x-2} \Rightarrow 4x - 8 = 1 \Rightarrow x = \frac{9}{4}; \qquad 3 = \frac{1}{y+1} \Rightarrow 3y + 3 = 1 \Rightarrow y = -\frac{2}{3}$$

The solution is $\left\{\left(\dfrac{9}{4}, -\dfrac{2}{3}\right)\right\}$.

90. Let $u = \dfrac{1}{x+1}$ and $v = \dfrac{1}{y-1}$. Then

$$\begin{cases} \dfrac{6}{x+1} + \dfrac{4}{y-1} = 7 \\ \dfrac{8}{x+1} + \dfrac{5}{y-1} = 9 \end{cases} \Rightarrow \begin{cases} 6u + 4v = 7 \\ 8u + 5v = 9 \end{cases}$$

$$D = \begin{vmatrix} 6 & 4 \\ 8 & 5 \end{vmatrix} = -2, D_u = \begin{vmatrix} 7 & 4 \\ 9 & 5 \end{vmatrix} = -1$$

$$D_y = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix} = -2$$

$$u = \frac{-1}{-2} = \frac{1}{2}, v = \frac{-2}{-2} = 1$$

(*continued on next page*)

(*continued*)

$$\frac{1}{2}=\frac{1}{x+1}\Rightarrow x+1=2\Rightarrow x=1$$
$$1=\frac{1}{y-1}\Rightarrow y-1=1\Rightarrow y=2$$

The solution is $\{(1, 2)\}$.

91. Let $u=\frac{1}{3^x}$ and $v=\frac{1}{4^y}$. Then

$$\begin{cases}\frac{1}{3^x}+\frac{4}{4^y}=25\\ \frac{2}{3^x}-\frac{1}{4^y}=14\end{cases}\Rightarrow\begin{cases}u+4v=25\\ 2u-\ \ v=14\end{cases}$$

$$D=\begin{vmatrix}1 & 4\\ 2 & -1\end{vmatrix}=-9, D_u=\begin{vmatrix}25 & 4\\ 14 & -1\end{vmatrix}=-81$$
$$D_y=\begin{vmatrix}1 & 25\\ 2 & 14\end{vmatrix}=-36$$
$$u=\frac{-81}{-9}=9, v=\frac{-36}{-9}=4$$
$$9=\frac{1}{3^x}\Rightarrow 3^x=\frac{1}{9}\Rightarrow x=-2$$
$$4=\frac{1}{4^y}\Rightarrow 4^y=\frac{1}{4}\Rightarrow y=-1$$

The solution is $\{(-2, -1)\}$.

92. We use the change of base formula along with the power rule of logarithms. See section 4.3.

a. Recall that $\log 8=\log\left(2^3\right)=3\log 2.$

$$\begin{vmatrix}\log_2 3 & \log_8 3\\ \log_3 4 & \log_3 4\end{vmatrix}$$
$$=\log_2 3\cdot\log_3 4-\log_8 3\cdot\log_3 4$$
$$=\frac{\log 3\cdot\log 4}{\log 2\cdot\log 3}-\frac{\log 3\cdot\log 4}{\log 8\cdot\log 3}$$
$$=\frac{\log 3\cdot\log 4}{\log 2\cdot\log 3}-\frac{\log 3\cdot\log 4}{3\log 2\cdot\log 3}$$
$$=\frac{\log 4}{\log 2}-\frac{\log 4}{3\log 2}$$
$$=\frac{3\log 4-\log 4}{3\log 2}=\frac{2\log 4}{3\log 2}$$
$$=\frac{2\log\left(2^2\right)}{3\log 2}=\frac{4\log 2}{3\log 2}=\frac{4}{3}$$

b. $\begin{vmatrix}\log_3 512 & \log_4 3\\ \log_3 8 & \log_4 9\end{vmatrix}$

$$=\log_3 512\cdot\log_4 9-\log_4 3\cdot\log_3 8$$
$$=\frac{\log 512\cdot\log 9}{\log 3\cdot\log 4}-\frac{\log 3\cdot\log 8}{\log 4\cdot\log 3}$$
$$=\frac{\log\left(2\cdot 16^2\right)\cdot 2\log 3}{\log 3\cdot\log 4}-\frac{3\log 2}{\log 4}$$
$$=\frac{\left(\log 2+2\log 16\right)\cdot 2}{2\log 2}-\frac{3\log 2}{2\log 2}$$
$$=\frac{\log 2+2\log\left(2^4\right)}{\log 2}-\frac{3}{2}$$
$$=\frac{\log 2+8\log 2}{\log 2}-\frac{3}{2}$$
$$=\frac{9\log 2}{\log 2}-\frac{3}{2}=9-\frac{3}{2}=\frac{15}{2}$$

93. The system has no solution when $D = 0$.

$$D=\begin{vmatrix}k & 3 & -1\\ 1 & 2 & 1\\ -k & 1 & 2\end{vmatrix}$$
$$=-1\begin{vmatrix}3 & -1\\ 1 & 2\end{vmatrix}+2\begin{vmatrix}k & -1\\ -k & 2\end{vmatrix}-1\begin{vmatrix}k & 3\\ -k & 1\end{vmatrix}$$
$$=-7+2(2k-k)-(k+3k)$$
$$=-7+2k-4k=-7-2k$$
$$-7-2k=0\Rightarrow k=-\frac{7}{2}$$

The system has no solution for $k=-\frac{7}{2}$.

94. $D=\begin{vmatrix}2 & a & 6\\ 1 & 2 & b\\ 1 & 1 & 3\end{vmatrix}=\begin{vmatrix}a & 6\\ 2 & b\end{vmatrix}-\begin{vmatrix}2 & 6\\ 1 & b\end{vmatrix}+3\begin{vmatrix}2 & a\\ 1 & 2\end{vmatrix}$

$$=(ab-12)-(2b-6)+3(4-a)$$
$$=ab-3a-2b+6$$

The system of equations has a unique solution when $D\neq 0$.

$$ab-3a-2b+6=0\Rightarrow ab-3a=2b-6\Rightarrow$$
$$a(b-3)=2(b-3)\Rightarrow a=2,\ b\neq 3$$

So, the system has a unique solution if $a\neq 2$ and $b\neq 3$.

If $a = 2$, then row 1 = 2 row 3, so there are infinitely many solutions.
If $a = b = 3$, then $D = 0$, but all rows are different, so there is no solution.

9.4 Maintaining Skills

You may want to refer to sections 1.2, 8.3, and 8.4.

95. $x^2 = 8y \Rightarrow \frac{1}{8}x^2 = y$

96. $(x+3)^2 = 4(y-1) \Rightarrow x^2 + 6x + 9 = 4y - 4 \Rightarrow$
$x^2 + 6x + 13 = 4y \Rightarrow \frac{1}{4}x^2 + \frac{3}{2}x + \frac{13}{4} = y$

97. $(x-2)^2 = 6(y-3) \Rightarrow x^2 - 4x + 4 = 6y - 18 \Rightarrow$
$x^2 - 4x + 22 = 6y \Rightarrow \frac{1}{6}x^2 - \frac{2}{3}x + \frac{11}{3} = y$

98. $(x-5)^2 = -2(y+7) \Rightarrow$
$x^2 - 10x + 25 = -2y - 14 \Rightarrow$
$x^2 - 10x + 39 = -2y \Rightarrow -\frac{1}{2}x^2 + 5x - \frac{39}{2} = y$

99. $-2 = a(-2)^2 \Rightarrow -2 = 4a \Rightarrow -\frac{1}{2} = a$

The function is $y = -\frac{1}{2}x^2$.

100. The formula for the x-value of the vertex gives
$x = -\frac{b}{2a} = 0 \Rightarrow b = 0$
Substituting (0, 0) and (2, −20) along with $b = 0$ in the general form of a quadratic equation gives
$-20 = a(2^2) + 0x + c \Rightarrow -20 = 4a + c$
$0 = a(0^2) + 0x + c \Rightarrow c = 0$
Since $c = 0$, $-20 = 4a + c \Rightarrow a = -5$.
The function is $y = -5x^2$.

101. The formula for the x-value of the vertex gives
$x = -\frac{b}{2a} = -3 \Rightarrow b = 6a$
Using (0, 10) gives
$10 = a(0^2) + b(0) + c \Rightarrow c = 10.$
Substituting (−3, −8), $b = 6a$, and $c = 10$ gives
$-8 = a(-3)^2 + 6a(-3) + 10 \Rightarrow$
$-8 = 9a - 18a + 10 \Rightarrow -18 = -9a \Rightarrow 2 = a \Rightarrow$
$b = 12$
The function is $y = 2x^2 + 12x + 10$.

102. Create a system of equations by substituting (1, 0), (−3, 0), and (0, 6) in the general form of a quadratic equation.

$$\begin{cases} 0 = a(1)^2 + b(1) + c \Rightarrow 0 = a + b + c \\ 0 = a(-3)^2 + b(-3) + c \Rightarrow 0 = 9a - 3b + c \\ 6 = a(0)^2 + b(0) + c \Rightarrow c = 6 \end{cases}$$

Substitute $c = 6$ into the first two equations, then solve the system made up of the first two equations.

$$\begin{cases} a + b + 6 = 0 \\ 9a - 3b + 6 = 0 \end{cases} \Rightarrow \begin{cases} a + b = -6 & (1) \\ 9a - 3b = -6 & (2) \end{cases}$$

Solve equation (1) for a in terms of b.
$a = -b - 6$
Substitute this expression for a in equation (2) and solve for b.
$9(-b-6) - 3b = -6 \Rightarrow -9b - 54 - 3b = -6 \Rightarrow$
$-12b = 48 \Rightarrow b = -4$
Substitute $b = -4$ in equation (1) and solve for a.
$a - 4 = -6 \Rightarrow a = -2$
The function is $y = -2x^2 - 4x - 6$.

103. $f(x) = 2x^2 + 4x - 6$

a. Opens up since $a > 0$.

b. $x = -\frac{b}{2a} = -\frac{4}{2(2)} = -1$
$f(-1) = 2(-1)^2 + 4(-1) - 6 = -8$
The vertex is (−1, −8).

c. The axis of symmetry is $x = -1$.

d. To find the x-intercepts, let $y = 0$ and solve for x.
$2x^2 + 4x - 6 = 0 \Rightarrow 2(x^2 + 2x - 3) = 0 \Rightarrow$
$(x+3)(x-1) = 0 \Rightarrow x = -3,\ x = 1$
The x-intercepts are −3 and 1.

e. To find the y-intercept, let $x = 0$ and solve for y.
$y = 2(0)^2 + 4(0) - 6 = -6$
The y-intercept is −6.

104. $f(x) = -2x^2 + 12x - 10$

a. Opens down since $a < 0$.

b. $x = -\frac{b}{2a} = -\frac{12}{2(-2)} = 3$
$f(3) = -2(3)^2 + 12(3) - 10 = 8$
The vertex is (3, 8).

c. The axis of symmetry is $x = 3$.

d. To find the x-intercepts, let $y = 0$ and solve for x.

$-2x^2 + 12x - 10 = 0 \Rightarrow$

$-2\left(x^2 - 6x + 5\right) = 0 \Rightarrow$

$(x-1)(x-5) = 0 \Rightarrow x = 1,\ x = 5$

The x-intercepts are 1 and 5.

e. To find the y-intercept, let $x = 0$ and solve for y.

$y = -2(0)^2 + 12(0) - 10 = -10$

The y-intercept is -10.

Chapter 9 Review Exercises

Basic Skills and Concepts

1. 1×4 **2.** 1×1

3. 3×2 **4.** 2×4

5. $a_{12} = -1, a_{14} = -4, a_{23} = 3, a_{21} = 5$

6. $\begin{bmatrix} -1 & 2 & 5 \\ 4 & 3 & 1 \end{bmatrix}$ **7.** $\left[\begin{array}{cc|c} 2 & -3 & 7 \\ 3 & 1 & 6 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & -2 & 2 \\ 5 & -3 & 1 & 8 \end{array}\right]$

9. Answers may vary.

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 2 & 0 & 3 & 4 \\ 1 & -2 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 1 & 7 \\ 2 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{2R_1 - R_2 \to R_2} \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & -4 & -1 & 10 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 1 & 2 & 1 \\ 0 & -4 & -1 & 10 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{7}(4R_2 + R_3) \to R_3} \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

10. $$\begin{bmatrix} 3 & -1 & 2 & 12 \\ 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 1 & 1 & -1 & 2 \\ 3 & -1 & 2 & 12 \end{bmatrix}$$

$$\xrightarrow[-3R_1 + R_3 \to R_3]{-(-R_1 + R_2) \to R_2} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 0 & 1 & 2 & 5 \\ 0 & -7 & -1 & -9 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{13}(7R_2 + R_3) \to R_3} \begin{bmatrix} 1 & 2 & 1 & 7 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

11. $$\begin{bmatrix} 3 & 1 & 3 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow[-3R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 3 & 1 \\ 0 & 4 & 6 & 1 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}(-2R_2 + R_3) \to R_2} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 4 & 6 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{6}(R_3 - 4R_2) \to R_3]{R_1 + R_2 \to R_1} \begin{bmatrix} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix}$$

$$\xrightarrow{R_1 + R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} \end{bmatrix}$$

12. $$\begin{bmatrix} 3 & 4 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & -2 & 1 \\ 3 & 4 & -4 & 2 \end{bmatrix}$$

$$\xrightarrow[3R_1 - R_3 \to R_3]{2R_1 - R_2 \to R_2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 6 & 1 \\ 0 & -1 & 10 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{16}(R_2 + R_3) \to R_3]{R_1 - R_2 \to R_1} \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & \frac{1}{8} \end{bmatrix}$$

$$\xrightarrow[R_2 - 6R_3 \to R_2]{R_1 + 4R_3 \to R_1} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{8} \end{bmatrix}$$

13. $\begin{cases} x+y-z=0 \\ 2x+y-2z=3 \\ 3x-2y+3z=9 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & -2 & 3 \\ 3 & -2 & 3 & 9 \end{array}\right]$

$\xrightarrow[3R_1-R_3\to R_3]{2R_1-R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 5 & -6 & -9 \end{array}\right]$

$\xrightarrow[-\frac{1}{6}(R_3-5R_2)\to R_3]{R_1-R_2\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

$\xrightarrow{R_1+R_3\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

The solution is $\{(2,-3,-1)\}$.

14. $\begin{cases} x-y+2z=1 \\ x+3y-z=6 \\ 2x+y-3z=1 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 3 & -1 & 6 \\ 2 & 1 & -3 & 1 \end{array}\right]$

$\xrightarrow[2R_1-R_3\to R_3]{\frac{1}{4}(R_2-R_1)\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -\frac{3}{4} & \frac{5}{4} \\ 0 & -3 & 7 & 1 \end{array}\right]$

$\xrightarrow[\frac{4}{19}(3R_2+R_3)\to R_3]{R_1+R_2\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{4} & \frac{9}{4} \\ 0 & 1 & -\frac{3}{4} & \frac{5}{4} \\ 0 & 0 & 1 & 1 \end{array}\right]$

$z=1;\ y-\frac{3}{4}(1)=\frac{5}{4}\Rightarrow y=2$

$x+\frac{5}{4}(1)=\frac{9}{4}\Rightarrow x=1$

The solution is $\{(1,2,1)\}$.

15. $\begin{cases} x-2y+3z=-2 \\ 2x-3y+z=9 \\ 3x-y+2z=5 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 2 & -3 & 1 & 9 \\ 3 & -1 & 2 & 5 \end{array}\right]$

$\xrightarrow[R_3-3R_1\to R_3]{R_2-2R_1\to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & -5 & 13 \\ 0 & 5 & -7 & 11 \end{array}\right]$

$\xrightarrow{\frac{1}{18}(R_3-5R_2)\to R_3} \left[\begin{array}{ccc|c} 1 & -2 & 3 & -2 \\ 0 & 1 & -5 & 13 \\ 0 & 0 & 1 & -3 \end{array}\right]$

$z=-3;\ y-5(-3)=13\Rightarrow y=-2$

$x-2(-2)+3(-3)=-2\Rightarrow x=3$

The solution is $\{(3,-2,-3)\}$.

16. $\begin{cases} 2x+y+z=7 \\ x+2y+z=3 \\ x+2y+2z=6 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 1 & 2 & 1 & 3 \\ 1 & 2 & 2 & 6 \end{array}\right]$

$\xrightarrow[2R_3-R_1\to R_3]{2R_2-R_1\to R_2} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 3 & 1 & -1 \\ 0 & 3 & 3 & 5 \end{array}\right]$

$\xrightarrow{\frac{1}{2}(R_3-R_2)\to R_3} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$

$\xrightarrow[\frac{1}{3}R_2\to R_2]{\frac{1}{2}R_1\to R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{7}{2} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & 3 \end{array}\right]$

$z=3;\ y+\frac{1}{3}(3)=-\frac{1}{3}\Rightarrow y=-\frac{4}{3}$

$x+\frac{1}{2}\left(-\frac{4}{3}\right)+\frac{1}{2}(3)=\frac{7}{2}\Rightarrow x=\frac{8}{3}$

The solution is $\left\{\left(\frac{8}{3},-\frac{4}{3},3\right)\right\}$.

17. $\begin{cases} x-2y-2z=11 \\ 3x+4y-z=-2 \\ 4x+5y+7z=7 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 3 & 4 & -1 & -2 \\ 4 & 5 & 7 & 7 \end{array}\right]$

$\xrightarrow[R_3-4R_1\to R_3]{\frac{1}{5}(R_2-3R_1)\to R_2} \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 2 & 1 & -7 \\ 0 & 13 & 15 & -37 \end{array}\right]$

$\xrightarrow[\frac{2}{17}(R_3-\frac{13}{2}R_2)\to R_3]{R_1+R_2\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 2 & 1 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\xrightarrow[\frac{1}{2}(R_2-R_3)\to R_2]{R_1+R_3\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$

The solution is $\{(5,-4,1)\}$.

18. $\begin{cases} x-y-9z=1 \\ 3x+2y-z=2 \\ 4x+3y+3z=0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -9 & 1 \\ 3 & 2 & -1 & 2 \\ 4 & 3 & 3 & 0 \end{array}\right]$

$\xrightarrow[R_3-4R_1\to R_3]{R_2-3R_1\to R_2} \left[\begin{array}{ccc|c} 1 & -1 & -9 & 1 \\ 0 & 5 & 26 & -1 \\ 0 & 7 & 39 & -4 \end{array}\right]$

$\xrightarrow{\frac{1}{13}(5R_3-7R_2)\to R_3} \left[\begin{array}{ccc|c} 1 & -1 & -9 & 1 \\ 0 & 5 & 26 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$

(*continued on next page*)

(*continued*)

$$\xrightarrow[\frac{1}{5}(R_2-26R_3)\to R_2]{R_1+9R_3\to R_1}\left[\begin{array}{ccc|c}1&-1&0&-8\\0&1&0&5\\0&0&1&-1\end{array}\right]$$

$$\xrightarrow{R_1+R_2\to R_1}\left[\begin{array}{ccc|c}1&0&0&-3\\0&1&0&5\\0&0&1&-1\end{array}\right]$$

The solution is $\{(-3,5,-1)\}$.

19. $\begin{cases}2x-\ y+3z=4\\ x+3y+3z=-2\\ 3x+2y-6z=6\end{cases}\Rightarrow\left[\begin{array}{ccc|c}2&-1&3&4\\1&3&3&-2\\3&2&-6&6\end{array}\right]$

$$\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{ccc|c}1&3&3&-2\\2&-1&3&4\\3&2&-6&6\end{array}\right]$$

$$\xrightarrow[R_3-3R_1\to R_3]{2R_1-R_2\to R_2}\left[\begin{array}{ccc|c}1&3&3&-2\\0&7&3&-8\\0&-7&-15&12\end{array}\right]$$

$$\xrightarrow{-\frac{1}{12}(R_2+R_3)\to R_3}\left[\begin{array}{ccc|c}1&3&3&-2\\0&7&3&-8\\0&0&1&-\frac{1}{3}\end{array}\right]$$

$$\xrightarrow[\frac{1}{7}(R_2-3R_3)\to R_2]{R_1-3R_3\to R_1}\left[\begin{array}{ccc|c}1&3&0&-1\\0&1&0&-1\\0&0&1&-\frac{1}{3}\end{array}\right]$$

$$\xrightarrow{R_1-3R_2\to R_1}\left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&-1\\0&0&1&-\frac{1}{3}\end{array}\right]$$

The solution is $\left\{\left(2,-1,-\frac{1}{3}\right)\right\}$.

20. $\begin{cases}x-\ y-9z=1\\ 3x+2y-\ z=2\\ 4x+3y+3z=0\end{cases}\Rightarrow\left[\begin{array}{ccc|c}1&-1&-9&1\\3&2&-1&2\\4&3&3&0\end{array}\right]$

$$\xrightarrow[R_3-4R_1\to R_3]{R_2-3R_1\to R_2}\left[\begin{array}{ccc|c}1&-1&-9&1\\0&5&26&-1\\0&7&39&-4\end{array}\right]$$

$$\xrightarrow{\frac{1}{13}(5R_3-7R_2)\to R_3}\left[\begin{array}{ccc|c}1&-1&-9&1\\0&5&26&-1\\0&0&1&-1\end{array}\right]$$

$$\xrightarrow[\frac{1}{5}(R_2-26R_3)\to R_2]{R_1+9R_3\to R_1}\left[\begin{array}{ccc|c}1&-1&0&-8\\0&1&0&5\\0&0&1&-1\end{array}\right]$$

$$\xrightarrow{R_1+R_2\to R_1}\left[\begin{array}{ccc|c}1&0&0&-3\\0&1&0&5\\0&0&1&-1\end{array}\right]$$

The solution is $\{(-3,5,-1)\}$.

21. $\begin{bmatrix}x-y&0\\1&x+y\end{bmatrix}=\begin{bmatrix}1&0\\1&3\end{bmatrix}\Rightarrow\begin{cases}x-y=1\\x+y=3\end{cases}\Rightarrow$

$2x=4\Rightarrow x=2,\ y=1$

22. $\begin{bmatrix}2x+3y&-1&-2\\0&1&x-y\\2&3x+4y&5\end{bmatrix}=\begin{bmatrix}4&-1&-2\\0&1&-3\\2&5&5\end{bmatrix}\Rightarrow$

$$\begin{cases}2x+3y=4\\ x-\ y=-3\\ 3x+4y=5\end{cases}\Rightarrow\left[\begin{array}{cc|c}2&3&4\\1&-1&-3\\3&4&5\end{array}\right]$$

$$\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{cc|c}1&-1&-3\\2&3&4\\3&4&5\end{array}\right]$$

$$\xrightarrow[\frac{1}{7}(R_3-3R_1)\to R_3]{\frac{1}{5}(R_2-2R_1)\to R_2}\left[\begin{array}{cc|c}1&-1&-3\\0&1&2\\0&1&2\end{array}\right]$$

$$\xrightarrow[R_2-R_3\to R_3]{R_1+R_2\to R_1}\left[\begin{array}{cc|c}1&0&-1\\0&1&2\\0&0&0\end{array}\right]\Rightarrow$$

$x=-1,\ y=2$

23. a. $\begin{bmatrix}1&2\\-3&4\end{bmatrix}+\begin{bmatrix}2&-3\\-5&6\end{bmatrix}=\begin{bmatrix}3&-1\\-8&10\end{bmatrix}$

b. $\begin{bmatrix}1&2\\-3&4\end{bmatrix}-\begin{bmatrix}2&-3\\-5&6\end{bmatrix}=\begin{bmatrix}-1&5\\2&-2\end{bmatrix}$

c. $2\begin{bmatrix}1&2\\-3&4\end{bmatrix}=\begin{bmatrix}2&4\\-6&8\end{bmatrix}$

d. $-3\begin{bmatrix}2&-3\\-5&6\end{bmatrix}=\begin{bmatrix}-6&9\\15&-18\end{bmatrix}$

e. $2\begin{bmatrix}1&2\\-3&4\end{bmatrix}-3\begin{bmatrix}2&-3\\-5&6\end{bmatrix}$

$$=\begin{bmatrix}2&4\\-6&8\end{bmatrix}-\begin{bmatrix}6&-9\\-15&18\end{bmatrix}=\begin{bmatrix}-4&13\\9&-10\end{bmatrix}$$

24. a. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ -2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 2 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ -2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

c. $2\begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ -2 & 0 & 3 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 4 & 4 \\ -4 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -3 \\ 3 & -3 & 0 \\ -3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -5 \\ 5 & 1 & 4 \\ -7 & 0 & 9 \end{bmatrix}$

25. $3A + 2B - 3X = 0 \Rightarrow 3\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} + 2\begin{bmatrix} 2 & -3 \\ -5 & 6 \end{bmatrix} = 3X \Rightarrow \begin{bmatrix} 3 & 6 \\ -9 & 12 \end{bmatrix} + \begin{bmatrix} 4 & -6 \\ -10 & 12 \end{bmatrix} = 3X \Rightarrow \begin{bmatrix} 7 & 0 \\ -19 & 24 \end{bmatrix} = 3X \Rightarrow$

$X = \begin{bmatrix} \frac{7}{3} & 0 \\ -\frac{19}{3} & 8 \end{bmatrix}$

26. $A - 2X + 2B = 0 \Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ -2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = 2X \Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ -2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 & -2 \\ 2 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix} = 2X \Rightarrow$

$\begin{bmatrix} 2 & 2 & -3 \\ 3 & 0 & 2 \\ -4 & 0 & 5 \end{bmatrix} = 2X \Rightarrow X = \begin{bmatrix} 1 & 1 & -\frac{3}{2} \\ \frac{3}{2} & 0 & 1 \\ -2 & 0 & \frac{5}{2} \end{bmatrix}$

27. a. $AB = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 0(-1)+1(-3) & 0(-1)+1(4) \\ 2(-1)+3(-3) & 2(-1)+3(4) \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -11 & 10 \end{bmatrix}$

b. $BA = \begin{bmatrix} -1 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1(0)-1(2) & -1(1)-1(3) \\ -3(0)+4(2) & -3(1)+4(3) \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 8 & 9 \end{bmatrix}$

28. a. $AB = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0(1)+1(3)+2(0) & 0(2)+1(-1)+2(4) \\ -1(1)+0(3)+1(0) & -1(2)+0(-1)+1(4) \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -1 & 2 \end{bmatrix}$

b. $BA = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(0)+2(-1) & 1(1)+2(0) & 1(2)+2(1) \\ 3(0)-1(-1) & 3(1)-1(0) & 3(2)-1(1) \\ 0(0)+4(-1) & 0(1)+4(0) & 0(2)+4(1) \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 3 & 5 \\ -4 & 0 & 4 \end{bmatrix}$

29. a. $AB = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1(2)+2(3)-1(1) \end{bmatrix} = \begin{bmatrix} 7 \end{bmatrix}$

b. $BA = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(2) & 2(-1) \\ 3(1) & 3(2) & 3(-1) \\ 1(1) & 1(2) & 1(-1) \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 3 & 6 & -3 \\ 1 & 2 & -1 \end{bmatrix}$

30. a. $AB = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1(1)+0(2) & 1(2)+0(-1) & 1(3)+0(2) & 1(4)+0(3) \\ 2(1)-1(2) & 2(2)-1(-1) & 2(3)-1(2) & 2(4)-1(3) \\ 3(1)-2(2) & 3(2)-2(-1) & 3(3)-2(2) & 3(4)-2(3) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 4 & 5 \\ -1 & 8 & 5 & 6 \end{bmatrix}$

b. BA is not defined.

31. $\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} A = \begin{bmatrix} 5 & 6 \\ 1 & -3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 5 & 6 \\ 1 & -3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}^{-1} = \frac{1}{-1-6}\begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$

$A = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

32. $B\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & -3 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 5 & 6 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}^{-1}$

$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}^{-1} = \frac{1}{-1-6}\begin{bmatrix} -1 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix}$

$B = \begin{bmatrix} 5 & 6 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{bmatrix} = \begin{bmatrix} \frac{23}{7} & \frac{4}{7} \\ -\frac{8}{7} & \frac{5}{7} \end{bmatrix}$

The results of exercises 31 and 32 show that $CA = BC$ does not imply $A = B$.

33. $\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}^{-1} = \frac{1}{10-12}\begin{bmatrix} 2 & -3 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{bmatrix}$

$A = \begin{bmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{bmatrix}$

34. $A\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}^{-1}$

$\begin{bmatrix} 5 & 3 \\ 4 & 2 \end{bmatrix}^{-1} = \frac{1}{10-12}\begin{bmatrix} 2 & -3 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} -1 & \frac{3}{2} \\ 2 & -\frac{5}{2} \end{bmatrix}$

The answers are the same.

35. a. $A^{-1} = \frac{1}{0-(-1)}\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$

b. $\left(A^2\right)^{-1} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}^2\right)^{-1} = \begin{bmatrix} -1 & 3 \\ -3 & 8 \end{bmatrix}^{-1}$

$= \frac{1}{-8+9}\begin{bmatrix} 8 & -3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 3 & -1 \end{bmatrix}$

c. $\left(A^{-1}\right)^2 = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 3 & -1 \end{bmatrix}$

d. $\left(A^2\right)\left(A^{-1}\right)^2 = \begin{bmatrix} -1 & 3 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $\left(A^{-1}\right)^2$ is the inverse of A^2.

36. a. $A^{-1} = \frac{1}{9-8}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

b. $\left(A^2\right)^{-1} = \left(\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}^2\right)^{-1} = \begin{bmatrix} 17 & 24 \\ 12 & 17 \end{bmatrix}^{-1}$

$= \frac{1}{17^2 - 12(24)}\begin{bmatrix} 17 & -24 \\ -12 & 17 \end{bmatrix}$

$= \begin{bmatrix} 17 & -24 \\ -12 & 17 \end{bmatrix}$

c. $\left(A^{-1}\right)^2 = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 17 & -24 \\ -12 & 17 \end{bmatrix}$

d. $\left(A^2\right)\left(A^{-1}\right)^2 = \begin{bmatrix} 17 & 24 \\ 12 & 17 \end{bmatrix}\begin{bmatrix} 17 & -24 \\ -12 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $\left(A^{-1}\right)^2$ is the inverse of A^2.

For exercises 37–40, we show that $AB = I$. You should also show that $BA = I$ in order to show that B is the inverse of A.

37. $AB = \begin{bmatrix} 7 & 6 \\ 6 & 5 \end{bmatrix}\begin{bmatrix} -5 & 6 \\ 6 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

38. $AB = \begin{bmatrix} -1 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

39. $AB = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}\left(\frac{1}{5}\right)\begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix} = \left(\frac{1}{5}\right)\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 4 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & -4 \\ 3 & 2 & -8 \\ -1 & 1 & 1 \end{bmatrix} = \left(\frac{1}{5}\right)\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

40. $AB = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ -2 & -1 & 4 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

41. $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}^{-1} = \dfrac{1}{12-2}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{3}{10} \end{bmatrix}$

42. $\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}^{-1} = \dfrac{1}{6+4}\begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{bmatrix}$

43. $\left[\begin{array}{rrr|rrr} 1 & 2 & -2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{rrr|rrr} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 5 & -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_2+2R_3 \to R_2]{R_1+R_3 \to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{2R_2+R_3 \to R_3} \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array}\right] \xrightarrow{R_1+R_3 \to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & 6 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array}\right]$

Thus, $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

44. $\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\2&2&4&0&1&0\\0&0&3&0&0&1\end{array}\right]\xrightarrow[\frac{1}{3}R_3\to R_3]{2R_1-R_2\to R_2}\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&2&-2&2&-1&0\\0&0&1&0&0&\frac{1}{3}\end{array}\right]\xrightarrow{R_1-R_2\to R_1}\left[\begin{array}{ccc|ccc}1&0&3&-1&1&0\\0&2&-2&2&-1&0\\0&0&1&0&0&\frac{1}{3}\end{array}\right]$

$\xrightarrow[\frac{1}{2}(R_2+2R_3)\to R_2]{R_1-3R_3\to R_1}\left[\begin{array}{ccc|ccc}1&0&0&-1&1&-1\\0&1&0&1&-\frac{1}{2}&\frac{1}{3}\\0&0&1&0&0&\frac{1}{3}\end{array}\right]$

Thus, $\begin{bmatrix}1&2&1\\2&2&4\\0&0&3\end{bmatrix}^{-1}=\begin{bmatrix}-1&1&-1\\1&-\frac{1}{2}&\frac{1}{3}\\0&0&\frac{1}{3}\end{bmatrix}$

45. $\begin{cases}x+3y=7\\2x+5y=4\end{cases}\Rightarrow\begin{bmatrix}1&3\\2&5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}7\\4\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&3\\2&5\end{bmatrix}^{-1}\begin{bmatrix}7\\4\end{bmatrix}=\begin{bmatrix}-5&3\\2&-1\end{bmatrix}\begin{bmatrix}7\\4\end{bmatrix}=\begin{bmatrix}-23\\10\end{bmatrix}$

The solution is $\{(-23, 10)\}$.

46. $\begin{cases}3x+5y=4\\2x+4y=5\end{cases}\Rightarrow\begin{bmatrix}3&5\\2&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3&5\\2&4\end{bmatrix}^{-1}\begin{bmatrix}4\\5\end{bmatrix}=\begin{bmatrix}2&-\frac{5}{2}\\-1&\frac{3}{2}\end{bmatrix}\begin{bmatrix}4\\5\end{bmatrix}=\begin{bmatrix}-\frac{9}{2}\\\frac{7}{2}\end{bmatrix}$

The solution is $\left\{\left(-\frac{9}{2},\frac{7}{2}\right)\right\}$.

47. $\begin{cases}x+3y+3z=3\\x+4y+3z=5\\x+3y+4z=6\end{cases}\Rightarrow\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\5\\6\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}^{-1}\begin{bmatrix}3\\5\\6\end{bmatrix}=\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}3\\5\\6\end{bmatrix}=\begin{bmatrix}-12\\2\\3\end{bmatrix}$

The solution is $\{(-12, 2, 3)\}$.

48. $\begin{cases}x+2y+3z=6\\2x+4y+5z=8\\3x+5y+6z=10\end{cases}\Rightarrow\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\8\\10\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}^{-1}\begin{bmatrix}6\\8\\10\end{bmatrix}=\begin{bmatrix}1&-3&2\\-3&3&-1\\2&-1&0\end{bmatrix}\begin{bmatrix}6\\8\\10\end{bmatrix}=\begin{bmatrix}2\\-4\\4\end{bmatrix}$

The solution is $\{(2, -4, 4)\}$.

49. $\begin{cases}x-y+z=3\\4x+2y=5\\7x-y-z=6\end{cases}\Rightarrow\begin{bmatrix}1&-1&1\\4&2&0\\7&-1&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\5\\6\end{bmatrix}\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&-1&1\\4&2&0\\7&-1&-1\end{bmatrix}^{-1}\begin{bmatrix}3\\5\\6\end{bmatrix}=\begin{bmatrix}\frac{1}{12}&\frac{1}{12}&\frac{1}{12}\\-\frac{1}{6}&\frac{1}{3}&-\frac{1}{6}\\\frac{3}{4}&\frac{1}{4}&-\frac{1}{4}\end{bmatrix}\begin{bmatrix}3\\5\\6\end{bmatrix}=\begin{bmatrix}\frac{7}{6}\\\frac{1}{6}\\2\end{bmatrix}$

The solution is $\left\{\left(\frac{7}{6},\frac{1}{6},2\right)\right\}$.

50. $\begin{cases} x+2y+4z=7 \\ 4x+3y-2z=6 \\ x \qquad -3z=4 \end{cases} \Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & -2 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & -2 \\ 1 & 0 & -3 \end{bmatrix}^{-1}\begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 & -6 & 16 \\ -10 & 7 & -18 \\ 3 & -2 & 5 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 91 \\ -100 \\ 29 \end{bmatrix}$

The solution is $\{(91, -100, 29)\}$.

51. $\begin{vmatrix} 2 & -5 \\ 3 & 4 \end{vmatrix} = 2(4) - (-5)(3) = 23$

52. $\begin{vmatrix} -1 & 4 \\ 11 & -3 \end{vmatrix} = (-1)(-3) - (4)(11) = -41$

53. $\begin{vmatrix} 12 & 21 \\ 4 & 7 \end{vmatrix} = 12(7) - 21(4) = 0$

54. $\begin{vmatrix} -7 & 9 \\ -4 & 5 \end{vmatrix} = (-7)(5) - 9(-4) = 1$

55. a. $A = \begin{bmatrix} 4 & -1 & 2 \\ -2 & -3 & 5 \\ 0 & 2 & -4 \end{bmatrix}$

$M_{12} = \begin{vmatrix} -2 & 5 \\ 0 & -4 \end{vmatrix} = (-2)(-4) - 5(0) = 8$ $\qquad M_{23} = \begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} = 4(2) - (-1)(0) = 8$

$M_{22} = \begin{vmatrix} 4 & 2 \\ 0 & -4 \end{vmatrix} = 4(-4) - 2(0) = -16$

b. $A_{12} = (-1)^{1+2} M_{12} = -8$ $\qquad A_{23} = (-1)^{2+3} M_{23} = -8$ $\qquad A_{22} = (-1)^{2+2} M_{22} = -16$

56. a. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 4(9) - 6(7) = -6$ $\qquad M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 1(8) - 2(7) = -6$

$M_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = 1(9) - 3(7) = -12$

b. $A_{12} = (-1)^{1+2} M_{12} = 6$
$A_{23} = (-1)^{2+3} M_{23} = 6$
$A_{22} = (-1)^{2+2} M_{22} = -12$

57. a. $\begin{vmatrix} 1 & -2 & 3 \\ 4 & -1 & -2 \\ -2 & 1 & 5 \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = 4(-1)^{2+1}\begin{vmatrix} -2 & 3 \\ 1 & 5 \end{vmatrix} - 1(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} - 2(-1)^{2+3}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}$

$= -4(-13) - 1(11) + 2(-3) = 35$

b. $\begin{vmatrix} 1 & -2 & 3 \\ 4 & -1 & -2 \\ -2 & 1 & 5 \end{vmatrix} = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} = 3(-1)^{1+3}\begin{vmatrix} 4 & -1 \\ -2 & 1 \end{vmatrix} - 2(-1)^{2+3}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + 5(-1)^{3+3}\begin{vmatrix} 1 & -2 \\ 4 & -1 \end{vmatrix}$

$= 3(2) + 2(-3) + 5(7) = 35$

58. a. $\begin{vmatrix} 1 & 4 & 3 \\ 6 & 8 & 10 \\ 2 & 5 & 4 \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = 6(-1)^{2+1}\begin{vmatrix} 4 & 3 \\ 5 & 4 \end{vmatrix} + 8(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + 10(-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}$

$= -6(1) + 8(-2) - 10(-3) = 8$

b. $\begin{vmatrix} 1 & 4 & 3 \\ 6 & 8 & 10 \\ 2 & 5 & 4 \end{vmatrix} a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} = 3(-1)^{1+3}\begin{vmatrix} 6 & 8 \\ 2 & 5 \end{vmatrix} + 10(-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} + 4(-1)^{3+3}\begin{vmatrix} 1 & 4 \\ 6 & 8 \end{vmatrix}$

$= 3(14) - 10(-3) + 4(-16) = 8$

59. Expand by the first column: $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + 0 + 1(-1)^{3+1}\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 2 + 4 = 6$

60. Expand by the first column:

$\begin{vmatrix} 2 & 3 & 4 \\ 2 & 5 & 9 \\ 2 & 7 & 16 \end{vmatrix} = 2(-1)^{1+1}\begin{vmatrix} 5 & 9 \\ 7 & 16 \end{vmatrix} + 2(-1)^{2+1}\begin{vmatrix} 3 & 4 \\ 7 & 16 \end{vmatrix} + 2(-1)^{3+1}\begin{vmatrix} 3 & 4 \\ 5 & 9 \end{vmatrix} = 2(17) - 2(20) + 2(7) = 8$

61. $D = \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} = -1,\ D_x = \begin{vmatrix} 11 & 3 \\ 4 & 1 \end{vmatrix} = -1,\ D_y = \begin{vmatrix} 5 & 11 \\ 2 & 4 \end{vmatrix} = -2$

$x = \frac{-1}{-1} = 1,\ y = \frac{-2}{-1} = 2$

The solution is $\{(1, 2)\}$.

62. $D = \begin{vmatrix} 2 & -7 \\ 5 & 6 \end{vmatrix} = 47,\ D_x = \begin{vmatrix} 13 & -7 \\ 9 & 6 \end{vmatrix} = 141,\ D_y = \begin{vmatrix} 2 & 13 \\ 5 & 9 \end{vmatrix} = -47$

$x = \frac{141}{47} = 3,\ y = \frac{-47}{47} = -1$

The solution is $\{(3, -1)\}$.

63. $D = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix} = -20,\ D_x = \begin{vmatrix} 14 & 1 & -1 \\ 16 & 3 & -1 \\ -10 & 1 & -3 \end{vmatrix} = -100,\ D_y = \begin{vmatrix} 3 & 14 & -1 \\ 1 & 16 & -1 \\ 1 & -10 & -3 \end{vmatrix} = -120,\ D_z = \begin{vmatrix} 3 & 1 & 14 \\ 1 & 3 & 16 \\ 1 & 1 & -10 \end{vmatrix} = -140$

$x = \frac{-100}{-20} = 5,\ y = \frac{-120}{-20} = 6,\ z = \frac{-140}{-20} = 7$

The solution is $\{(5, 6, 7)\}$.

64. $D = \begin{vmatrix} 2 & 3 & -2 \\ -3 & 5 & 4 \\ 3 & 7 & -6 \end{vmatrix} = -62,\ D_x = \begin{vmatrix} 0 & 3 & -2 \\ 9 & 5 & 4 \\ 4 & 7 & -6 \end{vmatrix} = 124,\ D_y = \begin{vmatrix} 2 & 0 & -2 \\ -3 & 9 & 4 \\ 3 & 4 & -6 \end{vmatrix} = -62,\ D_z = \begin{vmatrix} 2 & 3 & 0 \\ -3 & 5 & 9 \\ 3 & 7 & 4 \end{vmatrix} = 31$

$x = \frac{124}{-62} = -2,\ y = \frac{-62}{-62} = 1,\ z = \frac{31}{-62} = -\frac{1}{2}$

The solution is $\left\{\left(-2, 1, -\frac{1}{2}\right)\right\}$.

65. Expand by the third row:

$\begin{vmatrix} 1 & 2 & 4 \\ -3 & 5 & 7 \\ 1 & x & 4 \end{vmatrix} = 0 \Rightarrow 1(-1)^{3+1}\begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix} + x(-1)^{3+2}\begin{vmatrix} 1 & 4 \\ -3 & 7 \end{vmatrix} + 4(-1)^{3+3}\begin{vmatrix} 1 & 2 \\ -3 & 5 \end{vmatrix} = 0 \Rightarrow -6 - 19x + 4(11) = 0 \Rightarrow x = 2$

66. Expand by the second row:

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & x & 1 \\ 3 & 2 & x-1 \end{vmatrix} = 14 \Rightarrow 0 + x(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 3 & x-1 \end{vmatrix} + 1(-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = x(x-7) - 1(5) = 14 \Rightarrow x^2 - 7x - 5 = 14 \Rightarrow$$

$$x^2 - 7x - 19 = 0 \Rightarrow x = \frac{7 \pm \sqrt{49+76}}{2} = \frac{7 \pm 5\sqrt{5}}{2}$$

67. $$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 1 & -4 & 1 \end{vmatrix} = 0 \Rightarrow x(-1)^{1+1}\begin{vmatrix} 3 & 1 \\ -4 & 1 \end{vmatrix} + y(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = 0 \Rightarrow 7x - y - 11 = 0 \Rightarrow y = 7x - 11$$

Using the given points to find the slope and y-intercept, we have $m = \frac{3-(-4)}{2-1} = 7$ and $3 = 7(2) + b \Rightarrow b = -11$.

68. $$A = |D| = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 4 & 3 & 1 \\ 2 & 7 & 1 \end{vmatrix} = \left|\frac{1}{2}(16)\right| = 8 \text{ square units}$$

Applying the Concepts

69.

$$\begin{array}{l} \\ \text{Method 1} \\ \text{Method 2} \\ \text{Method 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{A} & \text{B} & \text{C} \end{array} \\ \begin{bmatrix} 4 & 8 & 2 \\ 5 & 7 & 1 \\ 5 & 4 & 8 \end{bmatrix} \end{array} \begin{bmatrix} 10 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 84 \\ 84 \\ 114 \end{bmatrix} \Rightarrow \text{Method 3 is the most profitable.}$$

70.

$$\begin{array}{l} \\ \text{Ardestanis} \\ \text{Barkley} \end{array} \begin{array}{c} \begin{array}{ccc} \text{M} & \text{F} & \text{C} \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} \begin{array}{cc} \text{Cals} & \text{Protein} \end{array} \\ \begin{bmatrix} 2200 & 50 \\ 1700 & 40 \\ 1500 & 30 \end{bmatrix} \end{array} = \begin{bmatrix} 2(2200)+3(1700)+1(1500) & 2(50)+3(40)+1(30) \\ 1(2200)+1(1700)+2(1500) & 1(50)+1(40)+2(30) \end{bmatrix} = \begin{array}{c} \begin{array}{cc} \text{Cals} & \text{Protein} \end{array} \\ \begin{bmatrix} 11{,}000 & 250 \\ 6900 & 150 \end{bmatrix} \end{array}$$

71. Let x = the speed of the plane. Let y = the velocity of the wind. Using Gauss-Jordan elimination, we have

$$\begin{cases} 3(x+y) = 1680 \\ 3.5(x-y) = 1680 \end{cases} \Rightarrow \begin{cases} 3x + 3y = 1680 \\ 3.5x - 3.5y = 1680 \end{cases} \Rightarrow$$

$$\left[\begin{array}{cc|c} 3 & 3 & 1680 \\ 3.5 & -3.5 & 1680 \end{array}\right]$$

$$\xrightarrow[\frac{1}{3.5}R_2 \to R_2]{\frac{1}{3}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 1 & 560 \\ 1 & -1 & 480 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}(R_1 - R_2) \to R_2]{\frac{1}{2}(R_1 + R_2) \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 520 \\ 0 & 1 & 40 \end{array}\right] \Rightarrow$$

$x = 520, y = 40$

The plane is traveling at 520 mph and the wind velocity is 40 mph.

72. Let x = Nertha's speed. Let y = Kristina's speed. Using Cramer's rule, we have

$$\begin{cases} 5x = 2 + 4y \\ 12y = 10 + 10x \end{cases} \Rightarrow \begin{cases} 5x - 4y = 2 \\ -10x + 12y = 10 \end{cases} \Rightarrow$$

$$D = \begin{vmatrix} 5 & -4 \\ -10 & 12 \end{vmatrix} = 20, D_x = \begin{vmatrix} 2 & -4 \\ 10 & 12 \end{vmatrix} = 64,$$

$$D_y = \begin{vmatrix} 5 & 2 \\ -10 & 10 \end{vmatrix} = 70$$

$$x = \frac{64}{20} = 3.2, y = \frac{70}{20} = 3.5$$

Nertha walks at 3.2 miles per hour, and Kristina walks at 3.5 miles per hour.

73. Let x = Andrew's amount. Let y = Bonnie's amount. Let z = Chauncie's amount. Using Gaussian elimination, we have

$$\begin{cases} x+y+z=320 \\ x=2z \\ y+z=x-20 \end{cases} \Rightarrow \begin{cases} x+y+z=320 \\ x \quad -2z=0 \\ -x+y+z=-20 \end{cases} \Rightarrow$$

$$\left|\begin{array}{ccc|c} 1 & 1 & 1 & 320 \\ 1 & 0 & -2 & 0 \\ -1 & 1 & 1 & -20 \end{array}\right|$$

$$\xrightarrow[\frac{1}{2}(R_1+R_3)\to R_3]{R_1-R_2\to R_2} \left|\begin{array}{ccc|c} 1 & 1 & 1 & 320 \\ 0 & 1 & 3 & 320 \\ 0 & 1 & 1 & 150 \end{array}\right|$$

$$\xrightarrow{\frac{1}{2}(R_2-R_3)\to R_3} \left|\begin{array}{ccc|c} 1 & 1 & 1 & 320 \\ 0 & 1 & 3 & 320 \\ 0 & 0 & 1 & 85 \end{array}\right| \Rightarrow$$

$z=85,\ y+3(85)=320 \Rightarrow y=65$

$x+65+85=320 \Rightarrow x=170$

Andrew has \$170, Bonnie has \$65, and Chauncie has \$85.

74. Let x = the number of men. Let y = the number of women. Using Cramer's rule, we have

$$\begin{cases} \frac{5}{8}y=x \\ \frac{7}{11}(y+1)=x+1 \end{cases} \Rightarrow \begin{cases} -8x+5y=0 \\ -11x+7y=4 \end{cases} \Rightarrow$$

$$D=\begin{vmatrix} -8 & 5 \\ -11 & 7 \end{vmatrix}=-1, D_x=\begin{vmatrix} 0 & 5 \\ 4 & 7 \end{vmatrix}=-20,$$

$$D_y=\begin{vmatrix} -8 & 0 \\ -11 & 4 \end{vmatrix}=-32$$

$$x=\frac{-20}{-1}=20,\ y=\frac{-32}{-1}=32$$

There were 20 men and 32 women.

75. Let x = the number of registered nurses. Let y = the number of licensed practical nurses. Let z = the number of nurse's aids. Using Gaussian elimination, we have

$$\begin{cases} 75x+20y+30z=4850 \\ 3x+\ \ 5y+\ \ 4z=530 \\ x+\quad y+\quad z=130 \end{cases} \Rightarrow$$

$$\left[\begin{array}{ccc|c} 75 & 20 & 30 & 4850 \\ 3 & 5 & 4 & 530 \\ 1 & 1 & 1 & 130 \end{array}\right]$$

$$\xrightarrow{R_1\leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 130 \\ 3 & 5 & 4 & 530 \\ 75 & 20 & 30 & 4850 \end{array}\right]$$

$$\xrightarrow[75R_1-R_3\to R_3]{\frac{1}{2}(R_2-3R_1)\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 130 \\ 0 & 1 & \frac{1}{2} & 70 \\ 0 & 55 & 45 & 4900 \end{array}\right]$$

$$\xrightarrow{\frac{2}{35}(R_3-55R_2)\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 130 \\ 0 & 1 & \frac{1}{2} & 70 \\ 0 & 0 & 1 & 60 \end{array}\right] \Rightarrow$$

$z=60,\ y+\frac{1}{2}(60)=70 \Rightarrow y=40,$

$x+60+40=130 \Rightarrow 30$

There are 30 registered nurses, 40 licensed practical nurses, and 60 nurses aides.

76. Let x = the medicine sales. Let y = the sales of nonmedical items. Let z = the sales of beer and cigarettes. Using Gauss-Jordan elimination

$$\begin{cases} x+y+z=18{,}500 \\ 0.08y+0.2z=1020 \\ x=y+z+3500 \end{cases} \Rightarrow \begin{cases} x+y+\quad z=18{,}500 \\ \quad 8y+20z=102{,}000 \\ x-y-\quad z=3500 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 18{,}500 \\ 0 & 8 & 20 & 102{,}000 \\ 1 & -1 & -1 & 3500 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}(R_1+R_3)\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11{,}000 \\ 0 & 8 & 20 & 102{,}000 \\ 1 & -1 & -1 & 3500 \end{array}\right]$$

$$\xrightarrow{R_1-R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11{,}000 \\ 0 & 8 & 20 & 102{,}000 \\ 0 & 1 & 1 & 7500 \end{array}\right]$$

$$\xrightarrow{\frac{1}{12}(R_2-8R_3)\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11{,}000 \\ 0 & 8 & 20 & 102{,}000 \\ 0 & 0 & 1 & 3500 \end{array}\right]$$

$$\xrightarrow{\frac{1}{8}(R_2-20R_3)\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11{,}000 \\ 0 & 1 & 0 & 4000 \\ 0 & 0 & 1 & 3500 \end{array}\right] \Rightarrow$$

$x=11{,}000,\ y=4000,\ z=3500$

The medicine sales were \$11,000; the nonmedical sales were \$4000; the beer and cigarette sales were \$3500.

Chapter 9 Practice Test A

1. $5\times 4,\ a_{43}=9$

2. $\left[\begin{array}{ccc|c} 7 & -3 & 9 & 5 \\ -2 & 4 & 3 & -12 \\ 8 & -5 & 1 & -9 \end{array}\right]$

3. $\begin{cases} 4x \quad\quad - z = -3 \\ x + 3y \quad = 9 \\ 2x + 7y + 5z = 8 \end{cases}$

4. $x - 5 = 5 \Rightarrow x = 10;\ y + 3 = 13 \Rightarrow y = 10;\ z = 0$
The solution is $\{(10, 10, 0)\}$.

5. $\begin{cases} x + 2y + z = 6 \\ x + y - z = 7 \\ 2x - y + 2z = -3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 1 & 1 & -1 & 7 \\ 2 & -1 & 2 & -3 \end{array}\right]$

$$\xrightarrow[\frac{1}{5}(2R_1 - R_3) \to R_3]{R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 3 \end{array}\right] \xrightarrow{\frac{1}{2}(R_2 - R_3) \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow[R_2 - 2R_3 \to R_2]{R_1 - R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The solution is $\{(2, 3, -2)\}$.

6. $\begin{cases} 2x + y - 4z = 6 \\ -x + 3y - z = -2 \\ 2x - 6y + 2z = 4 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & -4 & 6 \\ -1 & 3 & -1 & -2 \\ 2 & -6 & 2 & 4 \end{array}\right]$

$$\xrightarrow[R_1 - R_3 \to R_3]{R_1 + 2R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & 1 & -4 & 6 \\ 0 & 7 & -6 & 2 \\ 0 & 7 & -6 & 2 \end{array}\right] \xrightarrow[R_2 - R_3 \to R_3]{\frac{1}{14}(7R_1 - R_2) \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{11}{7} & \frac{20}{7} \\ 0 & 7 & -6 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{7}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{11}{7} & \frac{20}{7} \\ 0 & 1 & -\frac{6}{7} & \frac{2}{7} \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$x - \frac{11}{7}z = \frac{20}{7} \Rightarrow x = \frac{11}{7}z + \frac{20}{7}$
$y - \frac{6}{7}z = \frac{2}{7} \Rightarrow y = \frac{6}{7}z + \frac{2}{7}$
The solution is $\left\{\left(\frac{11}{7}z + \frac{20}{7}, \frac{6}{7}z + \frac{2}{7}, z\right)\right\}$.

7. $A - B = \begin{bmatrix} 5 & -2 \\ 4 & 0 \\ 7 & 6 \end{bmatrix} - \begin{bmatrix} -3 & -1 \\ 0 & -8 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ 4 & 8 \\ 11 & 0 \end{bmatrix}$

8. $AB = \begin{bmatrix} 3 & -7 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3(0) - 7(1) + 2(4) \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$

9. The product AB is not defined.

10. $2A = 2\begin{bmatrix} -3 & 1 & 0 \\ 5 & 7 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 2 & 0 \\ 10 & 14 & 4 \end{bmatrix}$

11. $A + BA = \begin{bmatrix} -3 & 1 & 0 \\ 5 & 7 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 8 & 2 \end{bmatrix}\begin{bmatrix} -3 & 1 & 0 \\ 5 & 7 & 2 \end{bmatrix}$
$= \begin{bmatrix} -3 & 1 & 0 \\ 5 & 7 & 2 \end{bmatrix} + \begin{bmatrix} 23 & 27 & 8 \\ -14 & 22 & 4 \end{bmatrix}$
$= \begin{bmatrix} 20 & 28 & 8 \\ -9 & 29 & 6 \end{bmatrix}$

12. $C^2 = \begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 25 \\ 0 & 16 \end{bmatrix}$

13. $C^{-1} = \begin{bmatrix} 1 & 5 \\ 0 & 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & -5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{5}{4} \\ 0 & \frac{1}{4} \end{bmatrix}$

14. $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & -1 \\ 3 & 1 & 5 \end{bmatrix}^{-1} = \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 3 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 3 & 1 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[3R_1 - R_3 \to R_3]{2R_1 - R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 3 & -5 & -1 & 2 & 0 \\ 0 & 5 & -8 & 0 & 3 & -1 \end{array}\right]$$

$$\xrightarrow[3R_3 - 5R_2 \to R_3]{5R_1 - R_2 \to R_1} \left[\begin{array}{ccc|ccc} 5 & 7 & 0 & 1 & 3 & 0 \\ 0 & 3 & -5 & -1 & 2 & 0 \\ 0 & 0 & 1 & 5 & -1 & -3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}(R_2 + 5R_3) \to R_2} \left[\begin{array}{ccc|ccc} 5 & 7 & 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 8 & -1 & -5 \\ 0 & 0 & 1 & 5 & -1 & -3 \end{array}\right]$$

(*continued on next page*)

(*continued*)

$$\xrightarrow{\frac{1}{5}(R_1-7R_2)\to R_1}\left[\begin{array}{ccc|ccc}1&0&0&-11&2&7\\0&1&0&8&-1&-5\\0&0&1&5&-1&-3\end{array}\right]$$

$$\begin{bmatrix}2&1&3\\1&2&-1\\3&1&5\end{bmatrix}^{-1}=\begin{bmatrix}-11&2&7\\8&-1&-5\\5&-1&-3\end{bmatrix}$$

15. $\begin{bmatrix}5&2\\3&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}32\\18\end{bmatrix}$ **16.** $\begin{cases}12x-3y=5\\-2x+7y=-9\end{cases}$

17. $\begin{bmatrix}1&5&-2\\4&-2&7\end{bmatrix}-5X=2\begin{bmatrix}2&5&-11\\18&8&11\end{bmatrix}\Rightarrow$

$\begin{bmatrix}1&5&-2\\4&-2&7\end{bmatrix}-5X=\begin{bmatrix}4&10&-22\\36&16&22\end{bmatrix}\Rightarrow$

$\begin{bmatrix}1&5&-2\\4&-2&7\end{bmatrix}-\begin{bmatrix}4&10&-22\\36&16&22\end{bmatrix}=5X\Rightarrow$

$\begin{bmatrix}-3&-5&20\\-32&-18&-15\end{bmatrix}=5X\Rightarrow$

$\begin{bmatrix}-\frac{3}{5}&-1&4\\-\frac{32}{5}&-\frac{18}{5}&-3\end{bmatrix}=X$

18. $\begin{vmatrix}\frac{1}{2}&-\frac{1}{4}\\\frac{1}{2}&\frac{3}{4}\end{vmatrix}=\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)-\left(-\frac{1}{4}\right)\left(\frac{1}{2}\right)=\frac{1}{2}$

19. Expand by the second row:

$\begin{vmatrix}1&3&5\\2&0&10\\-3&1&-15\end{vmatrix}$

$=2(-1)^{2+1}\begin{vmatrix}3&5\\1&-15\end{vmatrix}+0+10(-1)^{2+3}\begin{vmatrix}1&3\\-3&1\end{vmatrix}$

$=-2(-50)-10(10)=0$

20. $\begin{cases}2x-\ y+\ z=3\\x+\ y+\ z=6\\4x+3y-2z=4\end{cases}\Rightarrow D=\begin{vmatrix}2&-1&1\\1&1&1\\4&3&-2\end{vmatrix}$

$D_x=\begin{vmatrix}3&-1&1\\6&1&1\\4&3&-2\end{vmatrix}, D_y=\begin{vmatrix}2&3&1\\1&6&1\\4&4&-2\end{vmatrix},$

$D_z=\begin{vmatrix}2&-1&3\\1&1&6\\4&3&4\end{vmatrix}$

$x=\dfrac{\begin{vmatrix}3&-1&1\\6&1&1\\4&3&-2\end{vmatrix}}{\begin{vmatrix}2&-1&1\\1&1&1\\4&3&-2\end{vmatrix}}, y=\dfrac{\begin{vmatrix}2&3&1\\1&6&1\\4&4&-2\end{vmatrix}}{\begin{vmatrix}2&-1&1\\1&1&1\\4&3&-2\end{vmatrix}},$

$z=\dfrac{\begin{vmatrix}2&-1&3\\1&1&6\\4&3&4\end{vmatrix}}{\begin{vmatrix}2&-1&1\\1&1&1\\4&3&-2\end{vmatrix}}$

Chapter 9 Practice Test B

1. D **2.** D **3.** D

4. $x+3=9\Rightarrow x=6; y+4=2\Rightarrow y=-2; z=3$
The answer is C.

5. $\begin{cases}2x+y=15\\2y+z=25\\2z+x=26\end{cases}\Rightarrow\left[\begin{array}{ccc|c}2&1&0&15\\0&2&1&25\\1&0&2&26\end{array}\right]$

$\xrightarrow{R_1-2R_3\to R_3}\left[\begin{array}{ccc|c}2&1&0&15\\0&2&1&25\\0&1&-4&-37\end{array}\right]$

$\xrightarrow{\frac{1}{9}(R_2-2R_3)\to R_3}\left[\begin{array}{ccc|c}2&1&0&15\\0&2&1&25\\0&0&1&11\end{array}\right]\Rightarrow z=11$

The answer is C.

6. $\begin{cases}2x+y\qquad=17\\\quad y+2z=15\\x+\quad\ z=9\end{cases}\Rightarrow\left[\begin{array}{ccc|c}2&1&0&17\\0&1&2&15\\1&0&1&9\end{array}\right]$

$\xrightarrow{2R_3-R_1\to R_3}\left[\begin{array}{ccc|c}2&1&0&17\\0&1&2&15\\0&-1&2&1\end{array}\right]$

$\xrightarrow[\frac{1}{4}(R_2+R_3)\to R_3]{\frac{1}{2}(R_1+R_3)\to R_1}\left[\begin{array}{ccc|c}1&0&1&9\\0&1&2&15\\0&0&1&4\end{array}\right]$

$\xrightarrow[R_2-2R_3\to R_2]{R_1-R_3\to R_1}\left[\begin{array}{ccc|c}1&0&0&5\\0&1&0&7\\0&0&1&4\end{array}\right]\Rightarrow$

$x=5, y=7, z=4\Rightarrow 4(5)+3(7)+4=45$
The answer is D.

7. $\begin{bmatrix} -1 & 4 \\ 0 & 4 \\ 8 & -4 \end{bmatrix} - \begin{bmatrix} 7 & 2 \\ 17 & 4 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -8 & 2 \\ -17 & 0 \\ 6 & -6 \end{bmatrix}$

The answer is C.

8. $AB = \begin{bmatrix} -8 & 2 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -8(3) + 2(0) + 9(-3) \end{bmatrix}$

$= \begin{bmatrix} -51 \end{bmatrix}$. The answer is B.

9. AB is not defined. The answer is A.

10. $2A = 2\begin{bmatrix} 2 & 1 & -3 \\ -5 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -6 \\ -10 & 4 & 2 \end{bmatrix}$

The answer is B.

11. $A + BA = \begin{bmatrix} 2 & 1 & -3 \\ -5 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 7 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 & -3 \\ -5 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & -3 \\ -5 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -41 & 11 & 16 \\ -16 & 10 & -2 \end{bmatrix}$

$= \begin{bmatrix} -39 & 12 & 13 \\ -21 & 12 & -1 \end{bmatrix}$

The answer is C.

12. $C^2 = \begin{bmatrix} 5 & 4 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 29 & 20 \\ 5 & 4 \end{bmatrix}$

The answer is C.

13. $C^{-1} = \begin{bmatrix} 5 & 4 \\ 1 & 0 \end{bmatrix}^{-1} = -\frac{1}{4}\begin{bmatrix} 0 & -4 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{4} & -\frac{5}{4} \end{bmatrix}$

The answer is B.

14. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & -4 & 1 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow[3R_1 - R_3 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 4 & -1 & 3 & 0 & -1 \end{array}\right]$

$\xrightarrow{4R_2 - R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -11 & 4 & 1 \end{array}\right] \Rightarrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -11 & 4 & 1 \end{bmatrix}$

The answer is D.

15. A **16.** A

17. $\begin{bmatrix} -2 & -3 & 1 \\ -5 & 3 & -2 \end{bmatrix} - 3X = -5\begin{bmatrix} -1 & -2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} -2 & -3 & 1 \\ -5 & 3 & -2 \end{bmatrix} - 3X = \begin{bmatrix} 5 & 10 & 5 \\ -5 & 0 & -5 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} -2 & -3 & 1 \\ -5 & 3 & -2 \end{bmatrix} - \begin{bmatrix} 5 & 10 & 5 \\ -5 & 0 & -5 \end{bmatrix} = 3X \Rightarrow$

$3X = \begin{bmatrix} -7 & -13 & -4 \\ 0 & 3 & 3 \end{bmatrix} \Rightarrow$

$X = \begin{bmatrix} -\frac{7}{3} & -\frac{13}{3} & -\frac{4}{3} \\ 0 & 1 & 1 \end{bmatrix}$

The answer is B.

18. $\begin{vmatrix} -8 & 5 \\ -4 & -1 \end{vmatrix} = (-8)(-1) - 5(-4) = 28$

The answer is D.

19. $\begin{vmatrix} 2 & 3 & -2 \\ 3 & 0 & -3 \\ -3 & 0 & -5 \end{vmatrix} = 3(-1)^{1+2}\begin{vmatrix} 3 & -3 \\ -3 & -5 \end{vmatrix} = -3(-24) = 72$

The answer is C.

20. $\begin{cases} x + y + z = -6 \\ x - y + 3z = -22 \\ 2x + y + z = -10 \end{cases} \Rightarrow D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 4$

$D_x = \begin{vmatrix} -6 & 1 & 1 \\ -22 & -1 & 3 \\ -10 & 1 & 1 \end{vmatrix} = -16, D_y = \begin{vmatrix} 1 & -6 & 1 \\ 1 & -22 & 3 \\ 2 & -10 & 1 \end{vmatrix} = 12$

$D_z = \begin{vmatrix} 1 & 1 & -6 \\ 1 & -1 & -22 \\ 2 & 1 & -10 \end{vmatrix} = -20$

$x = \frac{-16}{4} = -4, y = \frac{12}{4} = 3, z = \frac{-20}{4} = -5$

The answer is B.

Cumulative Review Exercises (Chapters P–9)

1. $\sqrt{(-1-2)^2 + (y-(-3))^2} = 5 \Rightarrow$
$9 + (y+3)^2 = 25 \Rightarrow y^2 + 6y - 7 = 0 \Rightarrow$
$(y+7)(y-1) = 0 \Rightarrow y = -7$ or $y = 1$

2. $\frac{4}{x-1} - \frac{3}{x+2} = \frac{18}{(x+2)(x-1)} \Rightarrow$
$4(x+2) - 3(x-1) = 18 \Rightarrow x + 11 = 18 \Rightarrow x = 7$

3. $|2x-5| = 3 \Rightarrow 2x - 5 = 3$ or $2x - 5 = -3 \Rightarrow$
$x = 4$ or $x = 1$

4. $4x^2 = 8x - 13 \Rightarrow 4x^2 - 8x + 13 = 0 \Rightarrow$
$x = \dfrac{8 \pm \sqrt{64 - 4(4)(13)}}{2(4)} = \dfrac{8 \pm \sqrt{-144}}{8} = 1 \pm \dfrac{3}{2}i$

5. $\left(\dfrac{3x-1}{x+5}\right)^2 - 3\left(\dfrac{3x-1}{x+5}\right) - 28 = 0$
Let $u = \dfrac{3x-1}{x+5}$. Then we have $u^2 - 3u - 28 = 0 \Rightarrow$
$(u+4)(u-7) = 0 \Rightarrow u = -4$ or $u = 7$
$\dfrac{3x-1}{x+5} = -4 \Rightarrow 3x - 1 = -4x - 20 \Rightarrow x = -\dfrac{19}{7}$
$\dfrac{3x-1}{x+5} = 7 \Rightarrow 3x - 1 = 7x + 35 \Rightarrow x = -9$
The solution is $\left\{-\dfrac{19}{7}, -9\right\}$.

6. $\log_2 |x| + \log_2 |x+6| = 4 \Rightarrow$
$\log_2 (|x| \cdot |x+6|) = 4 \Rightarrow 2^4 = |x^2 + 6x| \Rightarrow$
$x^2 + 6x = 16$ or $x^2 + 6x = -16$
$x^2 + 6x - 16 = 0 \Rightarrow (x+8)(x-2) = 0 \Rightarrow$
$x = -8$ or $x = 2$
$x^2 + 6x + 16 = 0 \Rightarrow$
$x = \dfrac{-6 \pm \sqrt{36 - 64}}{2} = -3 \pm i\sqrt{7}$ (reject this)
The solution is $\{-8, 2\}$.

7. Solve $x + 2 = 0 \Rightarrow x = -2$ and
$2x - 1 = 0 \Rightarrow x = \dfrac{1}{2}$. The intervals to be tested are $(-\infty, -2), \left(-2, \dfrac{1}{2}\right)$, and $\left(\dfrac{1}{2}, \infty\right)$.

Interval	Test point	Value of $\dfrac{x+2}{2x-1}$	Result
$(-\infty, -2)$	–3	1/7	+
$(-2, 1/2)$	0	–2	–
$(1/2, \infty)$	1	3	+

The solution set is $(-\infty, -2) \cup \left(\dfrac{1}{2}, \infty\right)$.

8. Solve the associated equation:
$x^2 - 7x + 6 = 0 \Rightarrow (x-6)(x-1) = 0 \Rightarrow x = 6$ or $x = 1$. The intervals are $(-\infty, 1], [1, 6]$, and $[6, \infty)$.

Interval	Test point	Value of $x^2 - 7x + 6$	Result
$(-\infty, 1]$	0	6	+
$[1, 6]$	2	–4	–
$(6, \infty)$	7	6	+

The solution set is $[1, 6]$.

9. The factors of the constant term are $\{\pm 1, \pm 3\}$. The factors of the leading coefficient are $\{\pm 1, \pm 2, \pm 4\}$. The possible rational zeros are
$\left\{\pm 1, \pm\dfrac{1}{2}, \pm\dfrac{1}{4}, \pm 3, \pm\dfrac{3}{2}, \pm\dfrac{3}{4}\right\}$.

10. Using synthetic division, we have

$$\begin{array}{c|rrrr} \frac{1}{2} & 4 & 8 & -11 & 3 \\ & & 2 & 5 & -3 \\ \hline & 4 & 10 & -6 & 0 \end{array}$$

$4x^3 + 8x^2 - 11x + 3 = \left(x - \dfrac{1}{2}\right)(4x^2 + 10x - 6)$
The zeros of the depressed function $4x^2 + 10x - 6$ are also zeros of the original function.

$4x^2 + 10x - 6 = 0 \Rightarrow 2(x+3)(2x-1) = 0 \Rightarrow$
$x = -3$ or $x = \dfrac{1}{2}$.
So $\dfrac{1}{2}$ is a zero of multiplicity 2.

11. $h = kr^3 \Rightarrow 10{,}125 = k(15^3) \Rightarrow k = 3$
$h = 3(20^3) = 24{,}000$ horsepower

12. Let x = the number of acres to be annexed. Then
$0.12(400 + x) = 0.02(400) + 0.2x \Rightarrow$
$48 + 0.12x = 8 + 0.2x \Rightarrow x = 500$ square miles

13. Using substitution, we have
$\begin{cases} 3x + y = 2 \\ 4x + 5y = -1 \end{cases} \Rightarrow y = -3x + 2$
$4x + 5(-3x + 2) = -1 \Rightarrow -11x + 10 = -1 \Rightarrow$
$x = 1;\ y = -3(1) + 2 = -1$
The solution is $\{(1, -1)\}$.

14. Using elimination, we have

$$\begin{cases} 2x+y=5 \\ y^2-2y=-3x+5 \end{cases} \Rightarrow \begin{cases} 2x+\quad y=5 \\ 3x+y^2-2y=5 \end{cases} \Rightarrow$$

$$\begin{cases} -6x-\quad 3y=-15 \\ 6x+2y^2-4y=10 \end{cases} \Rightarrow 2y^2-7y=-5 \Rightarrow$$

$$2y^2-7y+5=0 \Rightarrow (y-1)(2y-5)=0 \Rightarrow$$

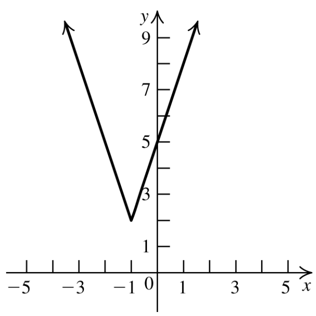
$$y=1 \text{ or } y=\frac{5}{2}$$

$$2x+1=5 \Rightarrow x=2$$

$$2x+\frac{5}{2}=5 \Rightarrow x=\frac{5}{4}$$

The solution is $\left\{(2,1),\left(\frac{5}{4},\frac{5}{2}\right)\right\}$.

15. Shift the graph of $f(x)=|x|$ one unit left, stretch by a factor of 3, then shift the resulting graph two units up.



16. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}^{-1} = \left[\begin{array}{rrr|rrr} 1 & 2 & -2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{R_1+R_2\to R_2} \left[\begin{array}{rrr|rrr} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 5 & -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_2+5R_3\to R_3]{R_1+R_3\to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 5 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array}\right]$$

$$\xrightarrow[\frac{1}{5}(R_2+2R_3)\to R_2]{R_1+R_3\to R_1} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & 6 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array}\right] \Rightarrow$$

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

17. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The solution is $\{(1, 1, -1)\}$.

18. a. $F(x)=(x+2)^2+3(x+2)-1=x^2+7x+9$

b. $F(4)=4^2+7(4)+9=53$

19. $y=\dfrac{x}{x+4}$. Switch the variables, and then solve for y to find $f^{-1}(x)$: $x=\dfrac{y}{y+4} \Rightarrow$

$$xy+4x=y \Rightarrow xy-y=-4x \Rightarrow$$

$$y(x-1)=-4x \Rightarrow y=-\frac{4x}{x-1} \Rightarrow$$

$$f^{-1}(x)=-\frac{4x}{x-1}$$

20. Domain: $(-\infty,-4)\cup(-4,\infty)$

Range: $(-\infty,1)\cup(1,\infty)$