Chapter 3 Polynomial and Rational Functions

3.1 Quadratic Functions

3.1 Practice Problems

1. Substitute 1 for h, −5 for k, 3 for x, and 7 for y in the standard form for a quadratic equation to solve for a:

   $$7 = a(3-1)^2 - 5 \Rightarrow 7 = 4a - 5 \Rightarrow a = 3$$

   The equation is $y = 3(x-1)^2 - 5.$

   Since $a = 3 > 0$, f has a minimum value of −5 at $x = 1$.

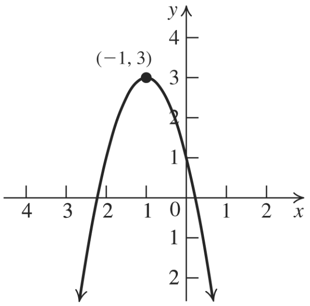
2. The graph of $f(x) = -2(x+1)^2 + 3$ is a parabola with $a = -2$, $h = -1$ and $k = 3$. Thus, the vertex is (−1, 3), and the maximum value of the function is 3. The parabola opens down because $a < 0$. Now, find the x-intercepts:

   $$0 = -2(x+1)^2 + 3 \Rightarrow 2(x+1)^2 = 3 \Rightarrow$$

   $$(x+1)^2 = \frac{3}{2} \Rightarrow x + 1 = \pm\sqrt{\frac{3}{2}} \Rightarrow x = \pm\sqrt{\frac{3}{2}} - 1 \Rightarrow$$

   $x \approx 0.22$ or $x \approx -2.22$. Next, find the y-intercept: $f(0) = -2(0+1)^2 + 3 = 1$

   Plot the vertex, the x-intercepts, and the y-intercept, and join them with a parabola.



3. The graph of $f(x) = 3x^2 - 3x - 6$ is a parabola with $a = 3$, $b = -3$ and $c = -6$. The parabola opens up because $a > 0$. Now, find the vertex:

   $$h = -\frac{b}{2a} = -\frac{-3}{2(3)} = \frac{1}{2}$$

   $$k = f(h) = f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) - 6 = -\frac{27}{4}$$

   Thus, the vertex (h, k) is $\left(\frac{1}{2}, -\frac{27}{4}\right)$, and the minimum value of the function is $-\frac{27}{4}$.

   Next, find the x-intercepts:

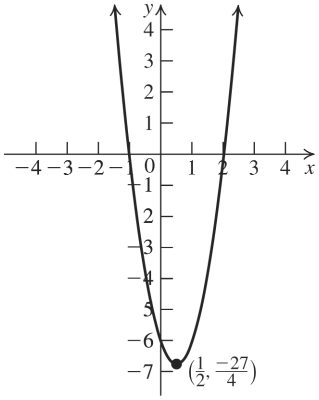
   $$3x^2 - 3x - 6 = 0 \Rightarrow 3\left(x^2 - x - 2\right) = 0 \Rightarrow$$

   $$(x-2)(x+1) = 0 \Rightarrow x = 2 \text{ or } x = -1$$

   Now, find the y-intercept:

   $$f(0) = 3(0)^2 - 3(0) - 6 = -6 .$$

   Thus, the intercepts are (−1, 0), (2, 0) and (0, −6). Use the fact that the parabola is symmetric with respect to its axis, $x = \frac{1}{2}$, to locate additional points. Plot the vertex, the x-intercepts, the y-intercept, and any additional points, and join them with a parabola.



4. The graph of $f(x) = 3x^2 - 6x - 1$ is a parabola with $a = 3$, $b = -6$ and $c = -1$. The parabola opens up because $a > 0$. Complete the square to write the equation in standard form:

   $$f(x) = 3x^2 - 6x - 1 = 3\left(x^2 - 2x\right) - 1$$
   $$= 3\left(x^2 - 2x + 1\right) - 1 - 3 = 3(x-1)^2 - 4$$

   Thus, the vertex is (1, −4). Next, find the x-intercepts:

   $$0 = 3(x-1)^2 - 4 \Rightarrow \frac{4}{3} = (x-1)^2 \Rightarrow$$

   $$\pm\frac{2\sqrt{3}}{3} = x - 1 \Rightarrow 1 \pm \frac{2\sqrt{3}}{3} = x \Rightarrow x \approx 2.15 \text{ or}$$

   $x \approx -0.15$. Now, find the y-intercept: $f(0) = 3(0)^2 - 6(0) - 1 = -1$. Use the fact that the parabola is symmetric with respect to its axis, $x = -1$, to locate additional points. Plot the vertex, the x-intercepts, the y-intercept, and any additional points, and join them with a parabola.

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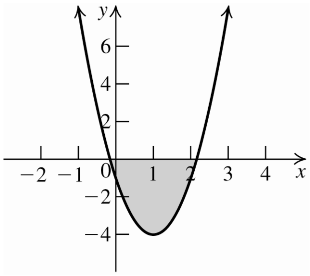


The graph of *f* is below the *x*-axis between the *x*-intercepts, so the solution set for $f(x) = 3x^2 - 6x - 1 \le 0$ is

$$\left[1 - \frac{2\sqrt{3}}{3},\ 1 + \frac{2\sqrt{3}}{3}\right] \text{ or } \left[\frac{3 - 2\sqrt{3}}{3}, \frac{3 + 2\sqrt{3}}{3}\right].$$

5. The graph of the parabola opens down, so $a < 0$.
The vertex $V(h, k)$ is in QII, so $h < 0$. Because $h = -\frac{b}{2a}$ and $a < 0$, we must have $b < 0$.
Otherwise, if b is positive, then $-b$ is negative, and $h = \frac{-b}{2a} = -\frac{b}{2a}$ is positive.
The *y*-intercept is positive, and the *y*-intercept is $a(0)^2 + b(0) + c = c$, so $c > 0$.

6. $h(t) = -\frac{g_E}{2}t^2 + v_0 t + h_0$

$g_E = 32\ \text{ft/s}^2$, $h_0 = 100$ ft, max height = 244 ft

a. Using the given values, we have

$$h(t) = -\frac{32}{2}t^2 + v_0 t + 100$$
$$= -16t^2 + v_0 t + 100$$

Using the formula for the vertex of a parabola gives $t = -\frac{v_0}{2(-16)} = \frac{v_0}{32}$. This is the time at which the maximum height $h(t) = 244$ ft is attained. Thus,

$$h(t) = 244 = h\left(\frac{v_0}{32}\right)$$
$$= -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) + 100$$
$$= -\frac{v_0^{\,2}}{64} + \frac{v_0^{\,2}}{32} + 100 = \frac{v_0^{\,2}}{64} + 100$$

Solving for v_0 yields

$$244 = \frac{v_0^{\,2}}{64} + 100 \Rightarrow 144 = \frac{v_0^{\,2}}{64} \Rightarrow$$
$$v_0^{\,2} = 144 \cdot 64 \Rightarrow v_0 = \sqrt{144 \cdot 64} = 12 \cdot 8 = 96$$

Thus, $h(t) = -16t^2 + 96t + 100$ feet.

b. Using the formula for the vertex of a parabola gives $t = -\frac{96}{2(-16)} = \frac{96}{32} = 3.$

The ball reached its highest point 3 seconds after it was released.

7. Let *x* = the length of the playground and *y* = the width of the playground.
Then $2(x + y) = 1000 \Rightarrow x + y = 500 \Rightarrow$
$y = 500 - x.$
The area of the playground is
$A(x) = xy = x(500 - x) = 500x - x^2.$
The vertex for the parabola is (h, k) where
$h = -\frac{500}{2(-1)} = 250$ and
$k = 500(250) - 250^2 = 62{,}500.$
Thus, the maximum area that can be enclosed is 62,500 ft^2. The playground is a square with side length 250 ft.

3.1 Basic Concepts and Skills

1. A point where the axis meets the parabola is called the <u>vertex</u>.

2. The vertex of the graph of $f(x) = -2(x + 3)^2 - 5$ is $\underline{(-3, -5)}$.

3. True. $a = -2 < 0$.

4. False. $a = 1 > 0$.

5. The *x*-coordinate of the vertex of the parabola in exercise 4, $f(x) = -2 - x + x^2$ is $\underline{\frac{1}{2}}$. Use the formula $h = -\frac{b}{2a}$, with $b = -1$ and $a = 1$.

6. True. The *x*-coordinate of the vertex of the parabola is $h = -\frac{-2}{2(1)} = 1$, so the *y*-coordinate of the vertex is $f(1)$.

7. True. The x-coordinate of the vertex of the parabola is $h = -\frac{b}{2a}$, so the y-coordinate of the vertex is $f\left(-\frac{b}{2a}\right)$.

8. True. If $a < 0$, then the parabola opens down and has a maximum.

9. f **10.** d **11.** a **12.** e

13. h **14.** b **15.** g **16.** c

17. Substitute −8 for y and 2 for x to solve for a: $-8 = a(2)^2 \Rightarrow -2 = a.$

The equation is $y = -2x^2$.

18. Substitute 3 for y and –3 for x to solve for a: $3 = a(-3)^2 \Rightarrow \frac{1}{3} = a.$

The equation is $y = \frac{1}{3}x^2$.

19. Substitute 20 for y and 2 for x to solve for a: $20 = a(2)^2 \Rightarrow 5 = a.$

The equation is $y = 5x^2$.

20. Substitute –6 for y and –3 for x to solve for a: $-6 = a(-3)^2 \Rightarrow -\frac{2}{3} = a.$

The equation is $y = -\frac{2}{3}x^2$.

21. Substitute 0 for h, 0 for k, 8 for y, and –2 for x in the standard form for a quadratic equation to solve for a: $8 = a(-2-0)^2 + 0 \Rightarrow 8 = 4a \Rightarrow 2 = a.$ The equation is $y = 2x^2$.

22. Substitute 2 for h, 0 for k, 3 for y, and 1 for x in the standard form for a quadratic equation to solve for a: $3 = a(1-2)^2 + 0 \Rightarrow 3 = a.$

The equation is $y = 3(x-2)^2$.

23. Substitute –3 for h, 0 for k, –4 for y, and –5 for x in the standard form for a quadratic equation to solve for a: $-4 = a(-5-(-3))^2 + 0 \Rightarrow -1 = a.$

The equation is $y = -(x+3)^2$.

24. Substitute 0 for h, 1 for k, 0 for y, and –1 for x in the standard form for a quadratic equation to solve for a: $0 = a(-1-0)^2 + 1 \Rightarrow -1 = a.$

The equation is $y = -x^2 + 1$.

25. Substitute 2 for h, 5 for k, 7 for y, and 3 for x in the standard form for a quadratic equation to solve for a: $7 = a(3-2)^2 + 5 \Rightarrow 2 = a.$

The equation is $y = 2(x-2)^2 + 5$.

26. Substitute –3 for h, 4 for k, 0 for y, and 0 for x in the standard form for a quadratic equation to solve for a: $0 = a(0-(-3))^2 + 4 \Rightarrow -\frac{4}{9} = a.$ The equation is $y = -\frac{4}{9}(x+3)^2 + 4.$

27. Substitute 2 for h, –3 for k, 8 for y, and –5 for x in the standard form for a quadratic equation to solve for a: $8 = a(-5-2)^2 - 3 \Rightarrow \frac{11}{49} = a.$

The equation is $y = \frac{11}{49}(x-2)^2 - 3.$

28. Substitute –3 for h, –2 for k, –8 for y, and 0 for x in the standard form for a quadratic equation to solve for a: $-8 = a(0-(-3))^2 - 2 \Rightarrow -\frac{2}{3} = a.$ The equation is $y = -\frac{2}{3}(x+3)^2 - 2.$

29. Substitute $\frac{1}{2}$ for h, $\frac{1}{2}$ for k, $-\frac{1}{4}$ for y, and $\frac{3}{4}$ for x in the standard form for a quadratic equation to solve for a: $-\frac{1}{4} = a\left(\frac{3}{4} - \frac{1}{2}\right)^2 + \frac{1}{2} \Rightarrow -12 = a.$

The equation is $y = -12\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}.$

30. Substitute $-\frac{3}{2}$ for h, $-\frac{5}{2}$ for k, $\frac{55}{8}$ for y, and 1 for x in the standard form for a quadratic equation to solve for a: $\frac{55}{8} = a\left(1 - \left(-\frac{3}{2}\right)\right)^2 - \frac{5}{2} \Rightarrow \frac{3}{2} = a.$

The equation is $y = \frac{3}{2}\left(x + \frac{3}{2}\right)^2 - \frac{5}{2}.$

31. The vertex is (–2, 0), and the graph passes through (0, 3). Substitute –2 for *h,* 0 for *k*, 3 for *y*, and 0 for *x* in the standard form for a quadratic equation to solve for *a*:

$$3 = a(0-(-2))^2 + 0 \Rightarrow \frac{3}{4} = a.$$

The equation is $y = \frac{3}{4}(x+2)^2$.

32. The vertex is (3, 0), and the graph passes through (0, 2). Substitute 3 for *h,* 0 for *k*, 2 for *y*, and 0 for *x* in the standard form for a quadratic equation to solve for *a*:

$$2 = a(0-3)^2 + 0 \Rightarrow \frac{2}{9} = a.$$

The equation is $y = \frac{2}{9}(x-3)^2$.

33. The vertex is (3, –1), and the graph passes through (5, 2). Substitute 3 for *h,* –1 for *k*, 2 for *y*, and 5 for *x* in the standard form for a quadratic equation to solve for *a*:

$$2 = a(5-3)^2 - 1 \Rightarrow \frac{3}{4} = a.$$

The equation is $y = \frac{3}{4}(x-3)^2 - 1$.

34. The vertex is (3, −1), and the graph passes through (6, 3). Substitute 3 for *h,* −1 for *k*, 3 for *y*, and 6 for *x* in the standard form for a quadratic equation to solve for *a*:

$$3 = a(6-3)^2 - 1 \Rightarrow \frac{4}{9} = a.$$

The equation is $y = \frac{4}{9}(x-3)^2 - 1$.

35. The graph opens up, so $a > 0$. The vertex is located in QIV, so $h > 0$. Because $h = -\frac{b}{2a}$ and $a > 0$, we must have $b < 0$. Otherwise, if *b* is positive, then −*b* is negative, and $h = \frac{-b}{2a} = -\frac{b}{2a}$ is negative.
The *y*-intercept is negative, and the *y*-intercept is $a(0)^2 + b(0) + c = c$, so $c < 0$.

36. The graph of the parabola opens down, so $a < 0$.
The vertex $V(h, k)$ lies on the negative *x*-axis, so $h < 0$ and $k = 0$. Because $h = -\frac{b}{2a}$ and $a < 0$, we must have $b < 0$.
Otherwise, if *b* is positive, then −*b* is negative, and $h = \frac{-b}{2a} = -\frac{b}{2a}$ is positive.
The *y*-intercept is negative, and the *y*-intercept is $a(0)^2 + b(0) + c = c$, so $c < 0$.

37. The graph opens up, so $a > 0$. The vertex is located on the positive *x*-axis, so $h > 0$ and $k = 0$. Because $h = -\frac{b}{2a}$ and $a > 0$, we must have $b < 0$. Otherwise, if *b* is positive, then −*b* is negative, and $h = \frac{-b}{2a} = -\frac{b}{2a}$ is negative.
The *y*-intercept is positive, and the *y*-intercept is $a(0)^2 + b(0) + c = c$, so $c > 0$.

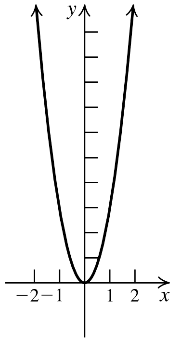
38. The graph of the parabola opens down, so $a < 0$.
The vertex $V(h, k)$ lies in QIV, so $h > 0$.
Because $h = -\frac{b}{2a}$ and $a < 0$, we must have $b > 0$. Otherwise, if *b* is negative, then −*b* is positive, and $h = \frac{-b}{2a} = -\frac{b}{2a}$ is negative.
The *y*-intercept is negative, and the *y*-intercept is $a(0)^2 + b(0) + c = c$, so $c < 0$.

39. The graph of the parabola opens down, so $a < 0$.
The vertex $V(h, k)$ is in QII, so $h < 0$. Because $h = -\frac{b}{2a}$ and $a < 0$, we must have $b < 0$.
Otherwise, if *b* is positive, then −*b* is negative, and $h = \frac{-b}{2a} = -\frac{b}{2a}$ is positive.
The *y*-intercept is negative, and the *y*-intercept is $a(0)^2 + b(0) + c = c$, so $c < 0$.

40. The graph opens up, so $a > 0$. The vertex is located in QIII, so $h < 0$. Because $h = -\frac{b}{2a}$ and $a > 0$, we must have $b > 0$. Otherwise, if *b* is negative, then −*b* is positive, and $h = \frac{-b}{2a} = -\frac{b}{2a}$ is positive.
The *y*-intercept is positive, and the *y*-intercept is $a(0)^2 + b(0) + c = c$, so $c > 0$.

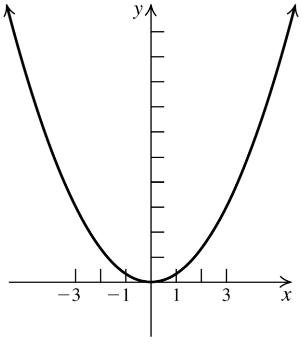
41. $f(x)=3x^2$

Stretch the graph of $y=x^2$ vertically by a factor of 3.



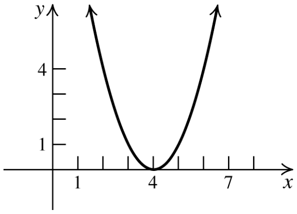
42. $f(x)=\frac{1}{3}x^2$

Compress the graph of $y=x^2$ vertically by a factor of $1/3$.



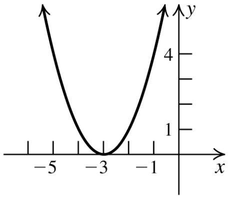
43. $g(x)=(x-4)^2$

Shift the graph of $y=x^2$ right four units.



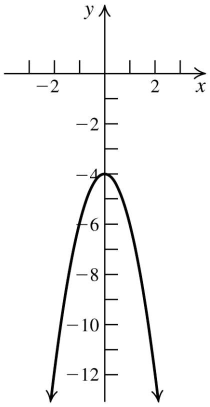
44. $g(x)=(x+3)^2$

Shift the graph of $y=x^2$ left three units.



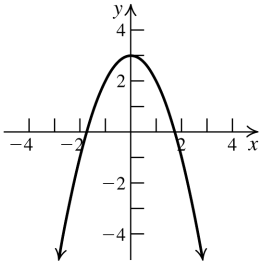
45. $f(x)=-2x^2-4$

Stretch the graph of $y=x^2$ vertically by a factor of 2, reflect the resulting graph in the *x*-axis, then shift the resulting graph down 4 units.



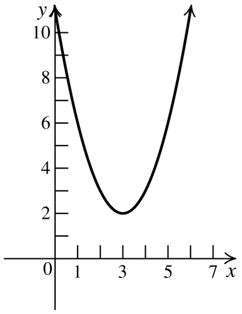
46. $f(x)=-x^2+3$

Reflect the graph of $y=x^2$ in the *x*-axis, then shift the resulting graph up 3 units.



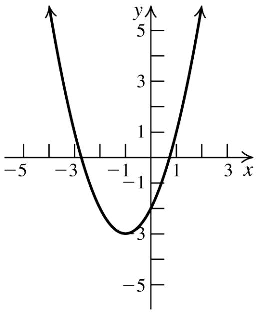
47. $g(x)=(x-3)^2+2$

Shift the graph of $y=x^2$ right three units, then shift the resulting graph up two units.



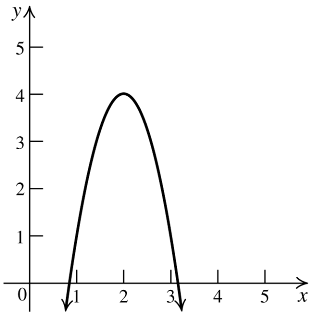
48. $g(x) = (x+1)^2 - 3$

Shift the graph of $y = x^2$ left one unit, then shift the resulting graph down three units.



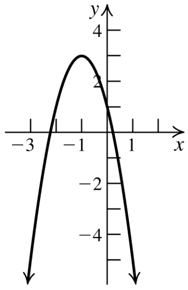
49. $f(x) = -3(x-2)^2 + 4$

Shift the graph of $y = x^2$ right two units, stretch the resulting graph vertically by a factor of 3, reflect the resulting graph about the *x*-axis, and then shift the resulting graph up four units.



50. $f(x) = -2(x+1)^2 + 3$

Shift the graph of $y = x^2$ left one unit, stretch the resulting graph vertically by a factor of 2, reflect the resulting graph about the *x*-axis, and then shift the resulting graph up three units.



51. Complete the square to write the equation in standard form: $y = x^2 + 4x \Rightarrow$

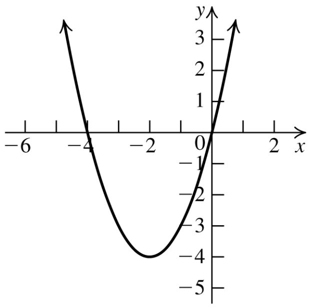
$y + 4 = x^2 + 4x + 4 \Rightarrow y = (x+2)^2 - 4.$ This is the graph of $y = x^2$ shifted two units left and four units down. The vertex is $(-2, -4)$. The axis of symmetry is $x = -2$. To find the *x*-intercepts, let $y = 0$ and solve

$0 = (x+2)^2 - 4 \Rightarrow (x+2)^2 = 4 \Rightarrow$

$x + 2 = \pm 2 \Rightarrow x = -4$ or $x = 0$

To find the *y*-intercept, let $x = 0$ and solve

$y = (0+2)^2 - 4 \Rightarrow y = 0.$

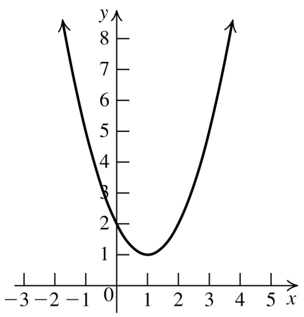


52. Complete the square to write the equation in standard form: $y = x^2 - 2x + 2 \Rightarrow$

$y + 1 = (x^2 - 2x + 1) + 2 \Rightarrow y = (x-1)^2 + 1.$

This is the graph of $y = x^2$ shifted one unit right and one unit up. The vertex is (1, 1). The axis of symmetry is $x = 1$. To find the *x*-intercepts, let $y = 0$ and solve

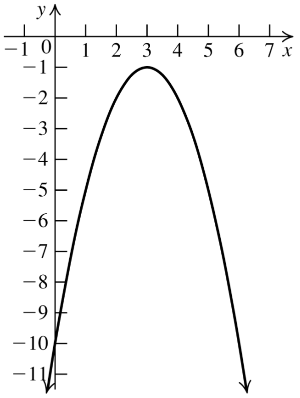
$0 = (x-1)^2 + 1 \Rightarrow (x-1)^2 = -1 \Rightarrow$ there is no *x*-intercept. To find the *y*-intercept, let $x = 0$ and solve $y = (0-1)^2 + 1 \Rightarrow y = 2.$



53. Complete the square to write the equation in standard form:

$$y = 6x - 10 - x^2 \Rightarrow y = -(x^2 - 6x + 10) \Rightarrow$$
$$y - 9 = -(x^2 - 6x + 9) - 10 \Rightarrow y = -(x-3)^2 - 1.$$

This is the graph of $y = x^2$ shifted three units right, reflected about the x-axis, and then shifted one unit down. The vertex is (3, –1). The axis of symmetry is $x = 3$. To find the x-intercepts, let $y = 0$ and solve $0 = -(x-3)^2 - 1 \Rightarrow -1 = (x-3)^2 \Rightarrow$ there is no x-intercept. To find the y-intercept, let $x = 0$ and solve $y = -(0-3)^2 - 1 \Rightarrow y = -10.$



54. Complete the square to write the equation in standard form:

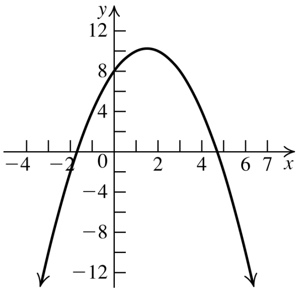
$$y = 8 + 3x - x^2 \Rightarrow y = -(x^2 - 3x - 8) \Rightarrow$$
$$y - \frac{9}{4} = -\left(x^2 - 3x + \frac{9}{4}\right) + 8 \Rightarrow$$
$$y = -\left(x - \frac{3}{2}\right)^2 + \frac{41}{4}.$$

This is the graph of $y = x^2$ shifted $3/2$ units right, reflected about the x-axis, and then shifted $\frac{41}{4}$ units up. The vertex is $\left(\frac{3}{2}, \frac{41}{4}\right)$.

The axis of symmetry is $x = \frac{3}{2}$. To find the x-intercepts, let $y = 0$ and solve

$$0 = -\left(x - \frac{3}{2}\right)^2 + \frac{41}{4} \Rightarrow \left(x - \frac{3}{2}\right)^2 = \frac{41}{4} \Rightarrow$$
$$x - \frac{3}{2} = \pm\frac{\sqrt{41}}{2} \Rightarrow x = \frac{3}{2} \pm \frac{1}{2}\sqrt{41}.$$ To find the y-intercept, let $x = 0$ and solve

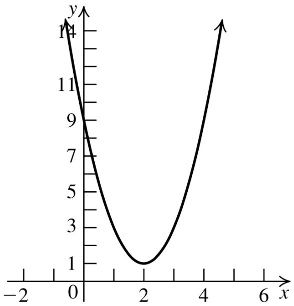
$$y = -\left(0 - \frac{3}{2}\right)^2 + \frac{41}{4} \Rightarrow y = 8.$$



55. Complete the square to write the equation in standard form:

$$y = 2x^2 - 8x + 9 \Rightarrow y = 2(x^2 - 4x) + 9 \Rightarrow$$
$$y + 8 = 2(x^2 - 4x + 4) + 9 \Rightarrow y = 2(x-2)^2 + 1.$$

This is the graph of $y = x^2$ shifted 2 units right, stretched vertically by a factor of 2, and then shifted one unit up. The vertex is (2, 1). The axis of symmetry is $x = 2$. To find the x-intercepts, let $y = 0$ and solve $0 = 2(x-2)^2 + 1 \Rightarrow -\frac{1}{2} = (x-2)^2 \Rightarrow$ there is no x-intercept. To find the y-intercept, let $x = 0$ and solve $y = 2(0-2)^2 + 1 \Rightarrow y = 9.$



56. Complete the square to write the equation in standard form:

$$y = 3x^2 + 12x - 7 \Rightarrow y = 3(x^2 + 4x) - 7 \Rightarrow$$
$$y + 12 = 3(x^2 + 4x + 4) - 7 \Rightarrow y = 3(x+2)^2 - 19.$$

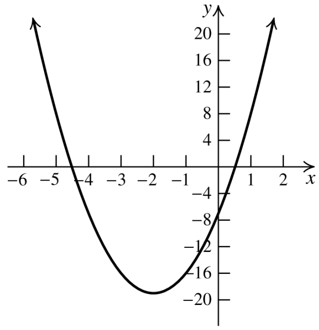
This is the graph of $y = x^2$ shifted 2 units left, stretched vertically by a factor of 3, and then shifted 19 units down. The vertex is (–2, –19). The axis of symmetry is $x = -2$. To find the x-intercepts, let $y = 0$ and solve

$$0 = 3(x+2)^2 - 19 \Rightarrow$$
$$\frac{19}{3} = (x+2)^2 \Rightarrow \pm\sqrt{\frac{19}{3}} = x + 2 \Rightarrow$$
$$\pm\frac{\sqrt{57}}{3} = x + 2 \Rightarrow -2 \pm \frac{1}{3}\sqrt{57} = x.$$

(*continued on next page*)

(*continued*)

To find the y-intercept, let $x = 0$ and solve $y = 3(0+2)^2 - 19 \Rightarrow y = -7.$



57. Complete the square to write the equation in standard form:

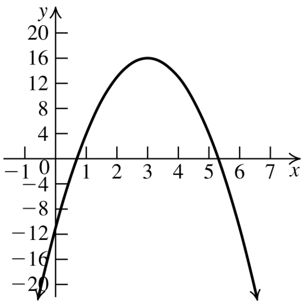
$y = -3x^2 + 18x - 11 \Rightarrow y = -3(x^2 - 6x) - 11 \Rightarrow$

$y - 27 = -3(x^2 - 6x + 9) - 11 \Rightarrow$

$y = -3(x-3)^2 + 16.$ This is the graph of $y = x^2$ shifted three units right, stretched vertically by a factor of three, reflected about the x-axis, and then shifted 16 units up. The vertex is (3, 16). The axis of symmetry is $x = 3$. To find the x-intercepts, let $y = 0$ and solve $y = 0 = -3(x-3)^2 + 16 \Rightarrow$

$\frac{16}{3} = (x-3)^2 \Rightarrow \pm\frac{4\sqrt{3}}{3} = x - 3 \Rightarrow$

$3 \pm \frac{4}{3}\sqrt{3} = x.$ To find the y-intercept, let $x = 0$ and solve $y = -3(0-3)^2 + 16 \Rightarrow y = -11.$



58. Complete the square to write the equation in standard form:

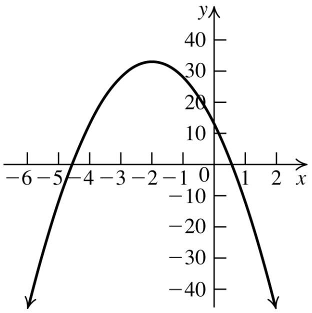
$y = -5x^2 - 20x + 13 \Rightarrow y = -5(x^2 + 4x) + 13 \Rightarrow$

$y - 20 = -5(x^2 + 4x + 4) + 13 \Rightarrow$

$y = -5(x+2)^2 + 33$.

This is the graph of $y = x^2$ shifted two units left, stretched vertically by a factor of five, reflected about the x-axis, and then shifted 33 units up. The vertex is (–2, 33). The axis is $x = -2$. To find the x-intercepts, let $y = 0$ and solve $0 = -5(x+2)^2 + 33 \Rightarrow \frac{33}{5} = (x+2)^2 \Rightarrow$

$\pm\sqrt{\frac{33}{5}} = x + 2 \Rightarrow -2 \pm \frac{1}{5}\sqrt{165} = x.$ To find the y-intercept, let $x = 0$ and solve $y = -5(0+2)^2 + 33 \Rightarrow y = 13.$



59. a. $a = 1 > 0$, so the graph opens up.

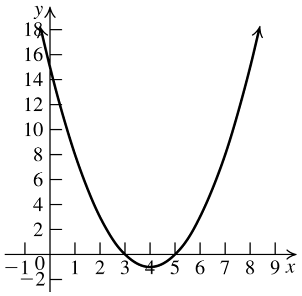
b. The vertex is

$$\left(-\frac{-8}{2(1)}, f\left(-\frac{-8}{2(1)}\right)\right) = (4, -1).$$

c. The axis of symmetry is $x = 4$.

d. To find the x-intercepts, let $y = 0$ and solve $0 = x^2 - 8x + 15 \Rightarrow 0 = (x-3)(x-5) \Rightarrow$ $x = 3$ or $x = 5.$ To find the y-intercept, let $x = 0$ and solve $y = 0^2 - 8(0) + 15 \Rightarrow$ $y = 15.$

e.



60. a. $a = 1 > 0$, so the graph opens up.

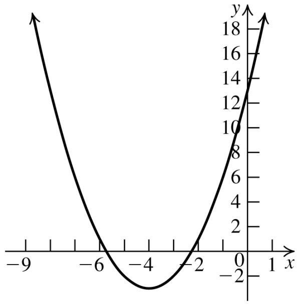
b. The vertex is

$$\left(-\frac{8}{2(1)}, f\left(-\frac{8}{2(1)}\right)\right) = (-4, -3).$$

c. The axis of symmetry is $x = -4$.

d. To find the x-intercepts, let $y = 0$ and solve $0 = x^2 + 8x + 13 \Rightarrow x = \dfrac{-8 \pm \sqrt{8^2 - 4(1)(13)}}{2(1)} \Rightarrow$ $x = \dfrac{-8 \pm \sqrt{12}}{2} = -4 \pm \sqrt{3}.$ To find the y-intercept, let $x = 0$ and solve $y = 0^2 + 8(0) + 13 = 13.$

e.



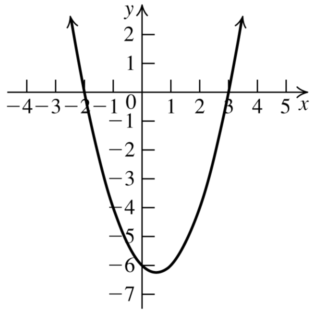
61. a. $a = 1 > 0$, so the graph opens up.

b. The vertex is $\left(-\dfrac{-1}{2(1)}, f\left(-\dfrac{-1}{2(1)}\right)\right) = \left(\dfrac{1}{2}, -\dfrac{25}{4}\right).$

c. The axis of symmetry is $x = \dfrac{1}{2}.$

d. To find the x-intercepts, let $y = 0$ and solve $0 = x^2 - x - 6 \Rightarrow 0 = (x-3)(x+2) \Rightarrow x = 3$ or $x = -2.$ To find the y-intercept, let $x = 0$ and solve $y = 0^2 - (0) - 6 \Rightarrow y = -6.$

e.



62. a. $a = 1 > 0$, so the graph opens up.

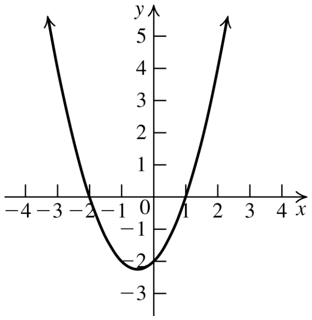
b. The vertex is $\left(-\dfrac{1}{2(1)}, f\left(-\dfrac{1}{2(1)}\right)\right) = \left(-\dfrac{1}{2}, -\dfrac{9}{4}\right).$

c. The axis of symmetry is $x = -\dfrac{1}{2}.$

d. To find the x-intercepts, let $y = 0$ and solve $0 = x^2 + x - 2 \Rightarrow 0 = (x-1)(x+2) \Rightarrow$ $x = 1$ or $x = -2.$

To find the y-intercept, let $x = 0$ and solve $y = 0^2 + (0) - 2 \Rightarrow y = -2.$

e.



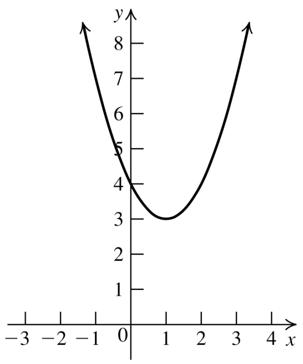
63. a. $a = 1 > 0$, so the graph opens up.

b. The vertex is $\left(-\dfrac{-2}{2(1)}, f\left(-\dfrac{-2}{2(1)}\right)\right) = (1, 3).$

c. The axis of symmetry is $x = 1.$

d. To find the x-intercepts, let $y = 0$ and solve $0 = x^2 - 2x + 4 \Rightarrow$ $x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} \Rightarrow$ $x = \dfrac{2 \pm \sqrt{-12}}{2} \Rightarrow$ there are no x-intercepts. To find the y-intercept, let $x = 0$ and solve $y = 0^2 - 2(0) + 4 \Rightarrow y = 4.$

e.



64. a. $a = 1 > 0$, so the graph opens up.

b. The vertex is $\left(-\dfrac{-4}{2(1)}, f\left(-\dfrac{-4}{2(1)}\right)\right) = (2, 1).$

c. The axis of symmetry is $x = 2.$

d. To find the x-intercepts, let $y = 0$ and solve

$0 = x^2 - 4x + 5 \Rightarrow$

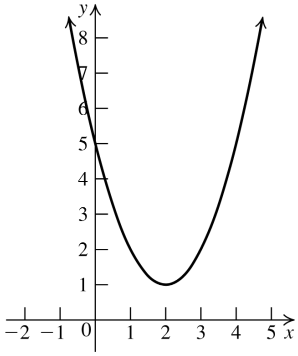
$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} \Rightarrow$

$x = \dfrac{4 \pm \sqrt{-4}}{2} \Rightarrow$ there are no x-intercepts.

To find the y-intercept, let $x = 0$ and solve

$y = 0^2 - 4(0) + 5 \Rightarrow y = 5.$

e.



65. a. $a = -1 < 0$, so the graph opens down.

b. The vertex is

$\left(-\dfrac{-2}{2(-1)}, f\left(-\dfrac{-2}{2(-1)}\right)\right) = (-1, 7).$

c. The axis of symmetry is $x = -1$.

d. To find the x-intercepts, let $y = 0$ and solve

$0 = 6 - 2x - x^2 \Rightarrow$

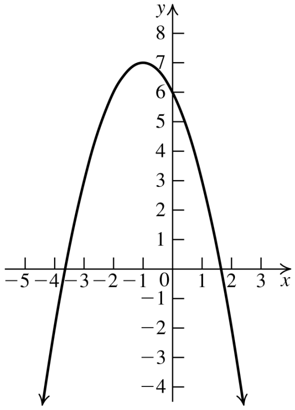
$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(-1)(6)}}{2(-1)} \Rightarrow x = \dfrac{2 \pm \sqrt{28}}{-2} \Rightarrow$

$x = -1 \pm \sqrt{7}.$

To find the y-intercept, let $x = 0$ and solve

$y = 6 - 2(0) - 0^2 \Rightarrow y = 6.$

e.



66. a. $a = -3 < 0$, so the graph opens down.

b. The vertex is

$\left(-\dfrac{5}{2(-3)}, f\left(-\dfrac{5}{2(-3)}\right)\right) = \left(\dfrac{5}{6}, \dfrac{49}{12}\right).$

c. The axis of symmetry is $x = \dfrac{5}{6}.$

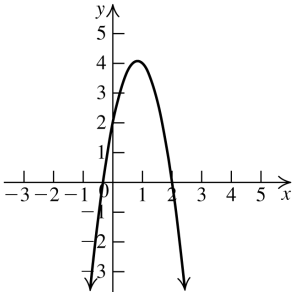
d. To find the x-intercepts, let $y = 0$ and solve

$0 = 2 + 5x - 3x^2 \Rightarrow 0 = -(3x^2 - 5x - 2) \Rightarrow$

$0 = -(3x + 1)(x - 2) \Rightarrow x = -\dfrac{1}{3}$ or $x = 2.$ To find the y-intercept, let $x = 0$ and solve

$y = 2 + 5(0) - 3(0)^2 \Rightarrow y = 2.$

e.



67. a. $a = 1 > 0$, so the graph opens up and has a minimum value. Find the minimum value by finding the vertex:

$\left(-\dfrac{-4}{2(1)}, f\left(-\dfrac{-4}{2(1)}\right)\right) = (2, -1)$

The minimum value is –1.

b. The range of f is $[-1, \infty).$

68. a. $a = -1 < 0$, so the graph opens down and has a maximum value. Find the maximum value by finding the vertex:

$\left(-\dfrac{6}{2(-1)}, f\left(-\dfrac{6}{2(-1)}\right)\right) = (3, 1)$

The maximum value is 1.

b. The range of f is $(-\infty, 1].$

69. a. $a = -1 < 0$, so the graph opens down and has a maximum value. Find the maximum value by finding the vertex:

$\left(-\dfrac{4}{2(-1)}, f\left(-\dfrac{4}{2(-1)}\right)\right) = (2, 0)$

The maximum value is 0.

b. The range of f is $(-\infty, 0].$

70. a. $a = 1 > 0$, so the graph opens up and has a minimum value. Find the minimum value by finding the vertex:

$$\left(-\frac{-6}{2(1)}, f\left(-\frac{-6}{2(1)}\right)\right) = (3,0)$$

The minimum value is 0.

b. The range of *f* is $[0, \infty)$.

71. a. $a = 2 > 0$, so the graph opens up and has a minimum value. Find the minimum value by finding the vertex:

$$\left(-\frac{-8}{2(2)}, f\left(-\frac{-8}{2(2)}\right)\right) = (2,-5)$$

The minimum value is –5.

b. The range of *f* is $[-5, \infty)$.

72. a. $a = 3 > 0$, so the graph opens up and has a minimum value. Find the minimum value by finding the vertex:

$$\left(-\frac{12}{2(3)}, f\left(-\frac{12}{2(3)}\right)\right) = (-2,-17)$$

The minimum value is –17.

b. The range of *f* is $[-17, \infty)$.

73. a. $a = -4 < 0$, so the graph opens down and has a maximum value. Find the maximum value by finding the vertex:

$$\left(-\frac{12}{2(-4)}, f\left(-\frac{12}{2(-4)}\right)\right) = \left(\frac{3}{2}, 16\right)$$

The maximum value is 16.

b. The range of *f* is $(-\infty, 16]$.

74. a. $a = -2 < 0$, so the graph opens down and has a maximum value. Find the maximum value by finding the vertex:

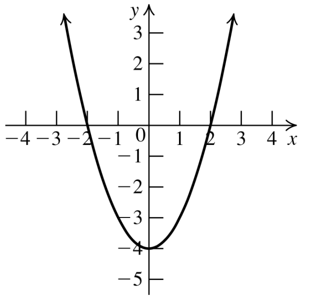
$$\left(-\frac{8}{2(-2)}, f\left(-\frac{8}{2(-2)}\right)\right) = (2,3)$$

The maximum value is 3.

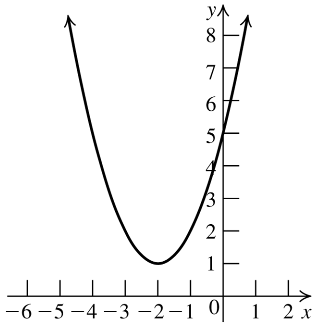
b. The range of *f* is $(-\infty, 3]$.

In exercises 75−80, the *x*-intercepts are the boundaries of the intervals.

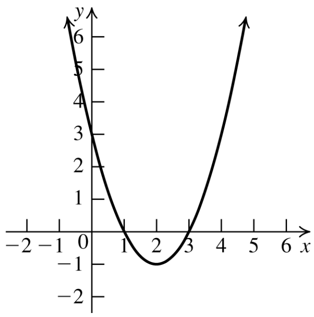
75. Solution: $[-2, 2]$



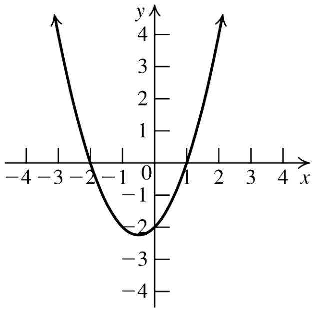
76. Solution: ∅



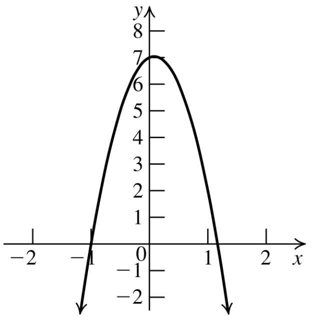
77. Solution:
$(-\infty, 1) \cup (3, \infty)$



78. Solution:
$(-\infty, -2) \cup$
$(1, \infty)$

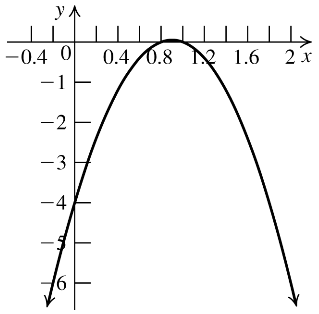


79. Solution:
$\left(-1, \frac{7}{6}\right)$



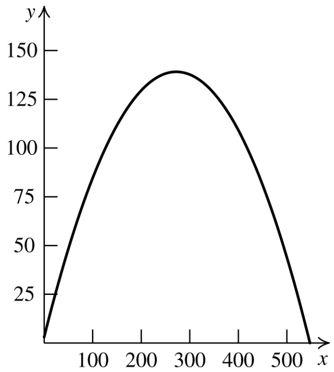
80. Solution:

$\left(-\infty, \frac{4}{5}\right] \cup [1, \infty)$



3.1 Applying the Concepts

81. $h(x) = -\frac{32}{132^2}x^2 + x + 3$



a. Using the graph, we see that the ball traveled approximately 550 ft horizontally.

b. Using the graph, we see that the ball went approximately 140 high.

c. $0 = -\frac{32}{132^2}x^2 + x + 3$

$$x = \frac{-1 \pm \sqrt{1^2 - 4\left(-\frac{32}{132^2}\right)(3)}}{2\left(-\frac{32}{132^2}\right)}$$

≈ -3 or 547

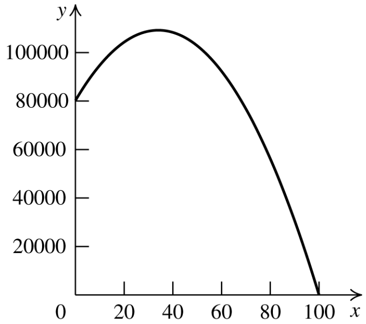
Thus, the ball traveled approximately 547 ft horizontally. The ball reached its maximum height at the vertex of the function, $\left(-\frac{b}{2a}, h\left(-\frac{b}{2a}\right)\right)$.

$$-\frac{b}{2a} = -\frac{1}{2\left(-32/\left(132^2\right)\right)} \approx 272.25$$

$$h(272.25) = -\frac{32}{132^2}(272.25)^2 + 272.25 + 3$$
$$\approx 139$$

The ball reached approximately 139 ft

82. $R(x) = -25x^2 + 1700x + 80,000$



a. Using the graph, we see that the maximum revenue from the apartments is approximately $110,000

b. Using the graph, we see that the maximum revenue is generated by about 35 $25-increases.

c. The maximum revenue is at the vertex of the function, $\left(-\frac{b}{2a}, R\left(-\frac{b}{2a}\right)\right)$.

$$-\frac{b}{2a} = -\frac{1700}{2(-25)} = 34$$

$$R(34) = -25(34)^2 + 1700(34) + 80,000$$
$$= 108,900$$

The maximum revenue of $108,900 is reached at 34 $25-increases.

83. The vertex of the function is $\left(-\frac{114}{2(-3)}, f\left(-\frac{114}{2(-3)}\right)\right) = (19, 1098).$

The revenue is at its maximum when $x = 19$.

84. $p = 200 - 4x \Rightarrow R(x) = 200x - 4x^2.$

The vertex of the revenue function is $\left(-\frac{200}{2(-4)}, f\left(-\frac{200}{2(-4)}\right)\right) = (25, 2500).$

The revenue is at its maximum when $x = 25$.

85. $\left(-\frac{-50}{2(1)}, f\left(-\frac{-50}{2(1)}\right)\right) = (25, -425).$

The total cost is minimum when $x = 25$.

86. $p = 100 - x \Rightarrow R(x) = 100x - x^2.$

Profit = revenue – cost

$\Rightarrow P(x) = \left(100x - x^2\right) - (50 + 2x) \Rightarrow$

$P(x) = -x^2 + 98x - 50.$

(*continued on next page*)

(*continued*)

Now find the vertex of the profit function:

$$\left(-\frac{98}{2(-1)}, f\left(-\frac{98}{2(-1)}\right)\right) = (49, 2351)$$

So the maximum profit occurs when $x = 49$. The maximum profit is \$2351.

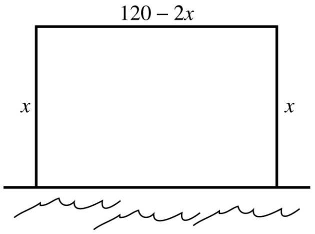
87. Let x = the length of the rectangle. Then $\frac{80-2x}{2} = 40 - x =$ the width of the rectangle. The area of the rectangle $= x(40-x)$ $= 40x - x^2$. Find the vertex to find the maximum value:

$$\left(-\frac{40}{2(-1)}, f\left(-\frac{40}{2(-1)}\right)\right) = (20, 400).$$

The rectangle with the maximum area is a square with sides of length 20 units. Its area is 400 square units.

88. The fence encloses three sides of the region. Let x = the width of the region. Then $120 - 2x$ = the length of the region. The area of the region =

$x(120-2x) = 120x - 2x^2$.

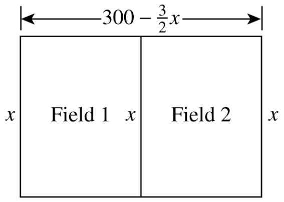


Find the vertex to find the maximum value:

$$\left(-\frac{120}{2(-2)}, f\left(-\frac{120}{2(-2)}\right)\right) = (30, 1800).$$

The maximum area that can be enclosed is 1800 square meters.

89. Let x = the width of the fields. Then, $\frac{600-3x}{2} = 300 - \frac{3}{2}x =$ the length of the two fields together. (Note that there is fencing between the two fields, so there are three "widths.")



The total area $= x\left(300 - \frac{3}{2}x\right) = 300x - \frac{3}{2}x^2$.

Find the vertex to find the dimensions and maximum value:

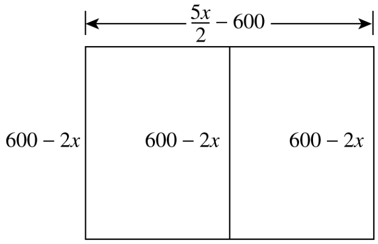
$$\left(-\frac{300}{2(-3/2)}, f\left(-\frac{300}{2(-3/2)}\right)\right) = (100, 15,000).$$

So the width of each field is 100 meters. The length of the two fields together is 300 – 1.5(100) = 150 meters, so the length of each field is 150/2 = 75 meters. The area of each field is 100(75) = 7500 square meters.

90. Let x = the amount of high fencing. Then $8x$ = the cost of the high fencing. The cost of the low fencing = $2400 - 8x$, and $\frac{2400-8x}{4} = 600 - 2x$ = the amount of low fencing. This is also the width of the enclosure. So

$$\frac{x - 2(600-2x)}{2} = \frac{5x - 1200}{2} = \frac{5}{2}x - 600 =$$

the length of the enclosure.



The area of the entire enclosure is

$$(600-2x)\left(\frac{5}{2}x - 600\right) = -5x^2 + 2700x - 360,000.$$

Use the vertex to find the dimensions and maximum area:

$$\left(-\frac{2700}{2(-5)}, f\left(-\frac{2700}{2(-5)}\right)\right) = (270, 4500)$$. So the area of the entire enclosure is 4500 square feet. There are 270 feet of high fencing, so the dimensions of the enclosure are $\frac{5(270)}{2} - 600 = 75$ feet by $600 - 2(270) = 60$ feet. The question asks for the dimensions and maximum area of each half of the enclosure, so each half has maximum area 2250 sq ft with dimensions 37.5 ft by 60 ft.

91. The yield per tree is modeled by the equation of a line passing through (26, 500) where the *x*-coordinate represents the number of trees planted, and the *y*-coordinate represents the number of apples per tree. The rate of change is –10; that is, for each tree planted the yield decreases by 10. So, the yield per tree is $y-500=-10(x-26)\Rightarrow y=-10x+760.$ Since there are *x* trees, the total yield = $x(-10x+760)=-10x^2+760x.$ Use the vertex to find the number of trees that will maximize the yield:

$$\left(-\frac{760}{2(-10)}, f\left(-\frac{760}{2(-10)}\right)\right)=(38{,}14{,}440).$$

So the maximum yield occurs when 38 trees are planted per acre.

92. Let *x* = the number of days. The original price is \$1.50 per pound and the price decreases \$0.02 per pound each day, so the price per pound is (1.5 – 0.02*x*). The weight of the steer after *x* days is 300 + 8*x*. So the selling price = number of pounds times price per pound $=(300+8x)(1.5-0.02x).$ The original cost of the steer is 1.5(300) = \$450, and the daily cost of the steer is *x*, so the total cost of the steer after *x* days is *x* + 450. The profit is selling price – cost

$$\begin{aligned}&=(300+8x)(1.5-0.02x)-(x+450)\\&=(-0.16x^2+6x+450)-(x+450)\\&=-0.16x^2+5x.\end{aligned}$$

The maximum profit occurs at the *x*-coordinate of the vertex:

$$\left(-\frac{5}{2(-0.16)}\right)=15.625.$$

The maximum profit occurs after 16 days.

93. If 20 students or less go on the trip, the cost is \$72 per students. If more than 20 students go on the trip, the cost is reduced by \$2 per the number of students over 20. So the cost per student is a piecewise function based on the number of students, *n*, going on the trip:

$$f(n)=\begin{cases}72 & \text{if } n\le 20\\ 72-2(n-20)=112-2n & \text{if } n>20\end{cases}$$

The total revenue is

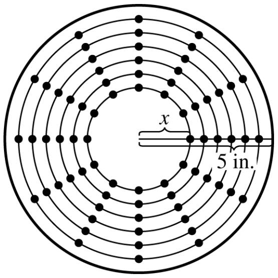
$$nf(n)=\begin{cases}72n & \text{if } n\le 20\\ n(112-2n)=112n-2n^2 & \text{if } n>20\end{cases}$$

The maximum revenue is either 1440 (the revenue if 20 students go on the trip) or the maximum of $112n-2n^2.$ Find this by using the vertex:

$$\left(-\frac{112}{(2)(-2)}, f\left(-\frac{112}{(2)(-2)}\right)\right)=(28{,}1568)$$

The maximum revenue is \$1568 when 28 students go on the trip.

94. Assume that *m* bytes per inch can be put on any track. Let *x* = the radius of the innermost track. Then the maximum number of bytes that can be put on the innermost track is $2\pi mx.$ So, each track will have $2\pi mx$ bytes.



The total number of bytes on the disk is the number of bytes on each track times the number of tracks per inch times the radius (in inches): $2\pi mx(p(5-x))=2\pi mp(5x-x^2)$ $=-2\pi mpx^2+10\pi mpx$. The maximum occurs at the *x*-coordinate of the vertex:

$$-\frac{b}{2a}=-\frac{10\pi mp}{2(-2\pi mp)}=\frac{10}{4}=2.5 \text{ inches.}$$

95. $h(t)=-\frac{g_M}{2}t^2+v_0t+h_0$

$g_M=1.6\,\text{m/s}^2$, $h_0=5$ m, max height = 25 m

a. Using the given values, we have

$$h(t)=-\frac{1.6}{2}t^2+v_0t+5=-0.8t^2+v_0t+5$$

Using the formula for the vertex of a parabola gives $t=-\frac{v_0}{2(-0.8)}=\frac{v_0}{1.6}$. This is the time at which the height *h*(*t*) = 25 m is attained. Thus,

$$\begin{aligned}h(t)=25&=h\left(\frac{v_0}{1.6}\right)\\&=-0.8\left(\frac{v_0}{1.6}\right)^2+v_0\left(\frac{v_0}{1.6}\right)+5\\&=-\frac{v_0^2}{3.2}+\frac{v_0^2}{1.6}+5=\frac{v_0^2}{3.2}+5\end{aligned}$$

(*continued on next page*)

(*continued*)

Solving for v_0 yields

$$25 = \frac{v_0{}^2}{3.2} + 5 \Rightarrow 20 = \frac{v_0{}^2}{3.2} \Rightarrow$$
$$v_0{}^2 = 64 \Rightarrow v_0 = 8$$

Thus, $h(t) = -0.8t^2 + 8t + 5.$

b. Using the formula for the vertex of a parabola gives $t = -\frac{8}{2(-0.8)} = \frac{8}{1.6} = 5.$

The ball reached its highest point 5 seconds after it was released.

96. $h(t) = -\frac{g_M}{2}t^2 + v_0 t + h_0$

$g_M = 1.6\,\text{m}/\text{s}^2$, $h_0 = 7$ m, max height = 52 m

a. Using the given values, we have

$$h(t) = -\frac{1.6}{2}t^2 + v_0 t + 7 = -0.8t^2 + v_0 t + 7$$

Using the formula for the vertex of a parabola gives $t = -\frac{v_0}{2(-0.8)} = \frac{v_0}{1.6}$. This is the time at which the height $h(t) = 52$ m is attained. Thus,

$$h(t) = 52 = h\left(\frac{v_0}{1.6}\right)$$
$$= -0.8\left(\frac{v_0}{1.6}\right)^2 + v_0\left(\frac{v_0}{1.6}\right) + 7$$
$$= -\frac{v_0{}^2}{3.2} + \frac{v_0{}^2}{1.6} + 7 = \frac{v_0{}^2}{3.2} + 7$$

Solving for v_0 yields

$$52 = \frac{v_0{}^2}{3.2} + 7 \Rightarrow 45 = \frac{v_0{}^2}{3.2} \Rightarrow$$
$$v_0{}^2 = 144 \Rightarrow v_0 = 12$$

Thus, $h(t) = -0.8t^2 + 12t + 7.$

b. Using the formula for the vertex of a parabola gives $t = -\frac{12}{2(-0.8)} = \frac{12}{1.6} = 7.5.$

The ball reached its highest point 7.5 seconds after it was released.

97. a. The maximum height occurs at the vertex: $\left(-\frac{64}{2(-16)}, f\left(-\frac{64}{2(-16)}\right)\right) = (2, 64)$, so the maximum height is 64 feet.

b. When the projectile hits the ground, $h = 0$, so solve

$0 = -16t^2 + 64t \Rightarrow -16t(t-4) = 0 \Rightarrow$
$t = 0$ or $t = 4$.

The projectile hits the ground at 4 seconds.

98. a. The maximum height occurs at the vertex: $\left(-\frac{64}{2(-16)}, f\left(-\frac{64}{2(-16)}\right)\right) = (2, 464).$

The maximum height is 464 feet.

b. When the projectile hits the ground, $h = 0$, so solve $0 = -16t^2 + 64t + 400 \Rightarrow$

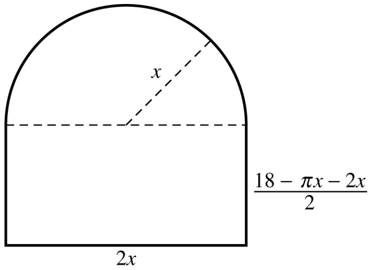
$-16(t^2 - 4t - 25) = 0 \Rightarrow t^2 - 4t - 25 = 0 \Rightarrow$

$$t = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-25)}}{2(1)} \Rightarrow t = \frac{4 \pm \sqrt{116}}{2} \Rightarrow$$

$t = 2 \pm \sqrt{29} \Rightarrow t \approx -3.39$ or $t \approx 7.39.$ Reject the negative solution (time cannot be negative). The projectile hits the ground at 7.39 seconds.

99. Let x = the radius of the semicircle. Then the length of the rectangle is $2x$. The circumference of the semicircle is πx, so the perimeter of the rectangular portion of the window is $18 - \pi x$.

The width of the rectangle = $\frac{18 - \pi x - 2x}{2}$.



The area of the semicircle is $\pi x^2/2$, and the area of the rectangle is $2x\left(\frac{18 - \pi x - 2x}{2}\right) =$

$18x - \pi x^2 - 2x^2$. So the total area is

$$18x - \pi x^2 - 2x^2 + \frac{\pi x^2}{2} = 18x - 2x^2 - \frac{\pi x^2}{2} \Rightarrow$$
$$18x - \left(2 + \frac{\pi}{2}\right)x^2.$$

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(*continued*)

The maximum area occurs at the *x*-coordinate of the vertex:

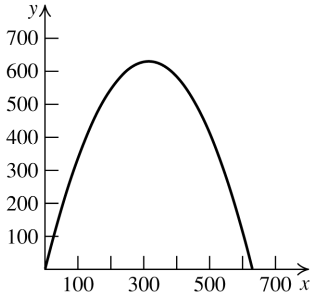
$$-\frac{18}{2\left(-2-\frac{\pi}{2}\right)}=-\frac{18}{-4-\pi}=\frac{18}{4+\pi}.$$

This is the radius of the semicircle. The length of the rectangle is $2\left(\frac{18}{4+\pi}\right)=\frac{36}{4+\pi}$ ft.

The width of the rectangle is

$$\frac{18-\pi\left(\frac{18}{4+\pi}\right)-2\left(\frac{18}{4+\pi}\right)}{2}=\frac{18}{4+\pi}\text{ ft.}$$

100. a.



b. The width of the arch is the difference between the two *x*-intercepts:

$0=-0.00635x^2+4x\Rightarrow$

$0=4x(-0.0015875x+1)\Rightarrow x=0$ or

$x\approx 629.92.$

The arch is 629.92 feet wide.

c. The maximum height occurs at the vertex of the arch. Since we know that the arch is 629.92 feet wide, the vertex occurs at $x=629.92/2=314.96.$

$$\begin{aligned}f(314.96)&=-0.00635(314.96)^2+4(314.96)\\&=629.92\text{ feet.}\end{aligned}$$

101. a. $V=\left(-\frac{1.18}{2(-0.01)},f\left(-\frac{1.18}{2(-0.01)}\right)\right)$
$=(59,36.81)$

b. $0=-0.01x^2+1.18x+2\Rightarrow$

$$\begin{aligned}x&=\frac{-1.18\pm\sqrt{1.18^2-4(-0.01)(2)}}{2(-0.01)}\\&=\frac{-1.18\pm\sqrt{1.4724}}{-0.02}\approx\frac{-1.18\pm1.21}{-0.02}\\&\approx-1.5\text{ or }119.5.\end{aligned}$$

Reject the negative answer, and round the positive answer to the nearest whole number. The ball hits the ground approximately 120 feet from the punter.

c. The maximum height occurs at the vertex. Since we know that the ball hits the ground at $x\approx 120,$ the vertex occurs at $x\approx 60.$

$f(60)=-0.01(60)^2+1.18(60)+2=36.8.$

The maximum height is approximately 37 ft.

d. The player is at *x* = 6 feet.

$f(6)=-0.01(6)^2+1.18(6)+2=8.72.$

The player must reach approximately 9 feet to block the ball.

e. $7=-0.01x^2+1.18x+2$

$$\begin{aligned}0&=-0.01x^2+1.18x-5\\x&=\frac{-1.18\pm\sqrt{1.18^2-4(-0.01)(-5)}}{2(-0.01)}\\&=\frac{-1.18\pm\sqrt{1.1924}}{-0.02}\Rightarrow\end{aligned}$$

$x\approx 4.4$ feet or $x\approx 113.6$ feet

102. The maximum height occurs at the vertex:

$$\left(-\frac{30}{2(-16)},f\left(-\frac{30}{2(-16)}\right)\right)=\left(\frac{15}{16},\frac{225}{16}\right).$$

The maximum height is approximately 14 feet, so it will never reach a height of 16 feet.

3.1 Beyond the Basics

103. $y=3(x+2)^2+3\Rightarrow y=3(x^2+4x+4)+3\Rightarrow$
$y=3x^2+12x+15$

104. Substitute the coordinates (1, 5) for *x* and *y* and (–3, –2) for *h* and *k* into the standard form $y=a(x-h)^2+k$ to solve for *a*:

$5=a(1+3)^2-2\Rightarrow 7=16a\Rightarrow\frac{7}{16}=a$ .The equation is $y=f(x)=\frac{7}{16}(x+3)^2-2\Rightarrow$

$y=\frac{7}{16}x^2+\frac{21}{8}x+\frac{31}{16}.$

105. The y-intercept is 4, so the graph passes through (0, 4). Substitute the coordinates (0, 4) for x and y and (1, –2) for h and k into the standard form $y = a(x-h)^2 + k$ to solve for a: $4 = a(0-1)^2 - 2 \Rightarrow 6 = a.$ The x-coordinate of the vertex is $1 = -\frac{b}{2a} = -\frac{b}{2(6)} \Rightarrow b = -12.$ The y-intercept = c, so the equation is $f(x) = 6x^2 - 12x + 4.$

106. The x-coordinate of the vertex of the graph is halfway between the x-intercepts: $\frac{2+6}{2} = 4 = h.$ Substitute the coordinates of one of the x-intercepts and the x-coordinate of the vertex into the standard form $y = a(x-h)^2 + k$ to find an expression relating a and k: $0 = a(2-4)^2 + k \Rightarrow -4a = k.$ Now substitute the coordinates of the y-intercept, the x-coordinate of the vertex, and the expression for k into the standard form $y = a(x-h)^2 + k$ to solve for a:

$24 = a(0-4)^2 - 4a \Rightarrow$

$24 = 16a - 4a \Rightarrow 2 = a.$ Use this value to find k: $k = -4(2) = -8.$ The equation is

$y = 2(x-4)^2 - 8 \Rightarrow y = 2x^2 - 16x + 24.$

107. The x-coordinate of the vertex is 3. Substitute the coordinates of the x-intercept and the x-coordinate of the vertex into the standard form $y = a(x-h)^2 + k$ to find an expression relating a and k:

$0 = a(7-3)^2 + k \Rightarrow -16a = k.$ Now substitute the coordinates of the y-intercept, the x-coordinate of the vertex, and the expression for k into the standard form $y = a(x-h)^2 + k$ to solve for a: $14 = a(0-3)^2 - 16a \Rightarrow -2 = a.$ Use this value to find k: $k = -16(-2) = 32.$ The equation is

$y = -2(x-3)^2 + 32 \Rightarrow y = -2x^2 + 12x + 14.$

108. First, write $y = x^2 + 2x + 2$ in standard form by completing the square:

$y + 1 = (x^2 + 2x + 1) + 2 \Rightarrow y = (x+1)^2 + 1.$

Move the curve three units to the right by subtracting 3 from $x + 1$; move the curve two units down by subtracting 2 from 1:

$y = (x+1-3)^2 + (1-2) = (x-2)^2 - 1 \Rightarrow$

$y = x^2 - 4x + 3.$

109. The x-coordinate of the vertex is $\frac{-2+6}{2} = 2.$ Substitute the coordinates of one of the x-intercepts and the x-coordinate of the vertex into the standard form $y = a(x-h)^2 + k$ to find an expression relating a and k:

$0 = a(-2-2)^2 + k \Rightarrow -16a = k.$

Now substitute the coordinates of the other x-intercept, the x-coordinate of the vertex, and the expression for k into the standard form $y = a(x-h)^2 + k$ to solve for a:

$0 = a(6-2)^2 - 16a \Rightarrow 1 = a.$ Use this value to find k: $-16(1) = -16 = k.$

So, one equation is

$y = (x-2)^2 - 16 = x^2 - 4x - 12.$ The graph of this equation opens upward. To find the equation of the graph that opens downward, multiply the equation by –1:

$y = -x^2 + 4x + 12.$

110. The x-coordinate of the vertex is $\frac{-3+5}{2} = 1.$ Substitute the coordinates of one of the x-intercepts and the x-coordinate of the vertex into the standard form $y = a(x-h)^2 + k$ to find an expression relating a and k:

$0 = a(-3-1)^2 + k \Rightarrow -16a = k.$ Now substitute the coordinates of the other x-intercept, the x-coordinate of the vertex, and the expression for k into the standard form $y = a(x-h)^2 + k$ to solve for a:

$0 = a(5-1)^2 - 16a \Rightarrow 1 = a.$ Use this value to find k: $-16(1) = -16 = k.$ So one equation is

$y = (x-1)^2 - 16 = x^2 - 2x - 15.$ The graph of this equation opens upward. To find the equation of the graph that opens downward, multiply the equation by –1:

$y = -x^2 + 2x + 15.$

111. The x-coordinate of the vertex is $\frac{-7-1}{2} = -4.$ Substitute the coordinates of one of the x-intercepts and the x-coordinate of the vertex into the standard form $y = a(x-h)^2 + k$ to find an expression relating a and k: $0 = a(-7+4)^2 + k \Rightarrow -9a = k.$ Now substitute the coordinates of the other x-intercept, the x-coordinate of the vertex, and the expression for k into the standard form $y = a(x-h)^2 + k$ to solve for a: $0 = a(-1+4)^2 - 9a \Rightarrow 1 = a.$ Use this value to find k: $-9(1) = -9 = k.$ So one equation is $y = (x+4)^2 - 9 = x^2 + 8x + 7.$ The graph of this equation opens upward. To find the equation of the graph that opens downward, multiply the equation by –1:

$y = -x^2 - 8x - 7.$

112. The x-coordinate of the vertex is $\frac{2+10}{2} = 6.$ Substitute the coordinates of one of the x-intercepts and the x-coordinate of the vertex into the standard form $y = a(x-h)^2 + k$ to find an expression relating a and k: $0 = a(2-6)^2 + k \Rightarrow -16a = k.$ Now substitute the coordinates of the other x-intercept, the x-coordinate of the vertex, and the expression for k into the standard form $y = a(x-h)^2 + k$ to solve for a: $0 = a(10-6)^2 - 16a \Rightarrow 1 = a.$ Use this value to find k: $-16(1) = -16 = k.$ So one equation is $y = (x-6)^2 - 16 = x^2 - 12x + 20.$ The graph of this equation opens upward. To find the equation of the graph that opens downward, multiply the equation by –1:

$y = -x^2 + 12x - 20.$

113.
$$\begin{aligned} f(a+1) &= 4(a+1) - (a+1)^2 \\ &= 4a + 4 - (a^2 + 2a + 1) = -a^2 + 2a + 3 \\ f(a-1) &= 4(a-1) - (a-1)^2 \\ &= 4a - 4 - (a^2 - 2a + 1) = -a^2 + 6a - 5 \end{aligned}$$
$$f(a+1) - f(a-1) = 0 \Rightarrow$$
$$(-a^2 + 2a + 3) - (-a^2 + 6a - 5) = 0 \Rightarrow$$
$$8 - 4a = 0 \Rightarrow a = 2$$

114. $(f \circ f)(x) = \left(x^2 + 1\right)^2 + 1 = x^4 + 2x^2 + 2$

3.1 Critical Thinking/Discussion/Writing

115.
$$f(h+p) = a(h+p)^2 + b(h+p) + c$$
$$f(h-p) = a(h-p)^2 + b(h+p) + c$$
$$f(h+p) = f(h-p) \Leftrightarrow$$
$$a(h+p)^2 + b(h+p) + c$$
$$= a(h-p)^2 + b(h-p) + c \Leftrightarrow$$
$$ah^2 + 2ahp + ap^2 + bh + bp + c$$
$$= ah^2 - 2ahp + ap^2 + bh - bp + c \Leftrightarrow$$
$4ahp = -2bp \Leftrightarrow 2ah = -b.$ (We can divide by p because $p \neq 0.$) Since $a \neq 0, 2ah = -b \Rightarrow$ $h = -\frac{b}{2a}.$

116. Write $y = 2x^2 - 8x + 9$ in standard form to find the axis of symmetry:
$$y = 2x^2 - 8x + 9 \Rightarrow y + 8 = 2(x^2 - 4x + 4) + 9 \Rightarrow$$
$$y = 2(x-2)^2 + 1.$$
The axis of symmetry is $x = 2.$
Using the results of exercise 115, we know that $f(2+p) = f(-1) = f(2-p).$
$2 + p = -1 \Rightarrow$
$p = -3,$ so $2 - p = 2 - (-3) = 5.$
The point symmetric to the point (–1, 19) across the axis of symmetry is (5, 19).

117. Answers will vary. If $a > 0$, then the parabola $f(x) = ax^2 + bx + c$ opens upward and the solution of the inequality is the portion of the graph that is above the x-axis. If the x-intercepts are represented by x_0 and x_1 with $x_0 \leq x_1,$ then the solution of the inequality is $(-\infty, x_0) \cup (x_1, \infty).$
If $a < 0$, then the parabola $f(x) = ax^2 + bx + c$ opens downward and the solution of the inequality is the portion of the graph that is below the x-axis. If the x-intercepts are represented by x_0 and x_1 with $x_0 \leq x_1,$ then the solution of the inequality is $(x_0, x_1).$

118. a.
$$\begin{aligned}(f\circ g)(x) &= f(mx+b) = a\left[(mx+b)-h\right]^2+k\\ &= a\left[(mx+b)^2-2h(mx+b)+h^2\right]+k\\ &= a\left[m^2x^2+2bmx+b^2-2hmx-2hb+h^2\right]+k\\ &= am^2x^2+2abmx+ab^2-2ahmx-2ahb+ah^2+k\\ &= am^2x^2+(2abm-2ahm)x+\left(ab^2-2ahb+ah^2+k\right)\end{aligned}$$

This is the equation of a parabola. The *x*-coordinate of the vertex is

$$-\frac{2abm-2ahm}{2am^2}=-\frac{2am(b-h)}{2am^2}=-\frac{b-h}{m}\text{ or }\frac{h-b}{m}.$$

The *y*-coordinate of the vertex is

$$\begin{aligned}&am^2\left(\frac{h-b}{m}\right)^2+(2abm-2ahm)\left(\frac{h-b}{m}\right)+\left(ab^2-2ahb+k\right)\\ &= a(h-b)^2+2a(b-h)(h-b)+\left(ab^2-2ahb+ah^2+k\right)\\ &= ah^2-2ahb+ab^2-2ab^2+4abh-2ah^2+ab^2-2ahb+ah^2+k\\ &= k\end{aligned}$$

The vertex is $\left(\frac{h-b}{m},\ k\right)$.

b.
$$\begin{aligned}(g\circ f)(x) &= g\left[a(x-h)^2+k\right]\\ &= m\left[a(x-h)^2+k\right]+b\\ &= m\left(ax^2-2ahx+ah^2+k\right)+b\\ &= max^2-2mahx+mah^2+mk+b\end{aligned}$$

This is the equation of a parabola. The *x*-coordinate of the vertex is $-\frac{-2mah}{2ma}=h.$

The *y*-coordinate of the vertex is $mah^2-2mah^2+mah^2+mk+b=mk+b.$

Thus, the vertex is $(h, mk+b)$.

119. If the discriminant equals zero, there is exactly one real solution. Thus, the vertex of $y=f(x)$ lies on the *x*-axis at $x=-\frac{b}{2a}.$ If the discriminant > 0, there are two unequal real solutions. This means that the graph of $y=f(x)$ crosses the *x*-axis in two places. If $a>0$, then the vertex lies below the *x*-axis and the parabola crosses the *x*-axis; if $a<0$, then the vertex lies above the *x*-axis and the parabola crosses the *x*-axis. If the discriminant < 0, there are two nonreal complex solutions. If $a>0$, then the vertex lies above the *x*-axis and the parabola does not cross the *x*-axis (it opens upward); if $a<0$, then the vertex lies below the *x*-axis and the parabola does not cross the *x*-axis (it opens downward).

120. Let $g(x)=x^2$ and $k(x)=x-h.$ Then $f(x)=g(k(x))=g(x-h)=(x-h)^2,$ which is a horizontal translation of *g*.

3.1 Maintaining Skills

121. $-5^0=-\left(5^0\right)=-1$

122. $(-4)^{-2}=\frac{1}{(-4)^2}=\frac{1}{16}$

123. $-4^{-2}=-(4)^{-2}=-\frac{1}{(4)^2}=-\frac{1}{16}$

124.
$$\begin{aligned}\left(-2x^2\right)\left(-3x^4\right)&=(-2)(-3)\left(x^2\right)\left(x^4\right)\\&=6x^{2+4}=6x^6\end{aligned}$$

125. $x^7\cdot x^{-7}=x^{7+(-7)}=x^0=1$

126.
$$\begin{aligned}\left(2x^2\right)(3x)\left(-x^3\right)&=(2)(3)(-1)\left(x^2\right)(x)\left(x^3\right)\\&=-6x^{2+1+3}=-6x^6\end{aligned}$$

127. $x^2\left(3-\frac{3}{4}\right)=x^2\left(\frac{12}{4}-\frac{3}{4}\right)=\frac{9}{4}x^2$

128.
$$\begin{aligned}x^4\left(1+\frac{3}{x^2}-\frac{5}{x^3}\right)&=x^4(1)+x^4\left(\frac{3}{x^2}\right)+x^4\left(-\frac{5}{x^3}\right)\\&=x^4+3x^2-5x\end{aligned}$$

129. $4x^2-9=(2x+3)(2x-3)$

130. $x^2+6x+9=(x+3)^2$

131. $15x^2+11x-12=(3x+4)(5x-3)$

132. $14x^2-3x-2=(2x-1)(7x+2)$

133. $$\begin{aligned}x^2(x-1)-4(x-1)&=(x^2-4)(x-1)\\&=(x+2)(x-2)(x-1)\end{aligned}$$

134. $$\begin{aligned}&9x^2(2x+7)-25(2x+7)\\&=(9x^2-25)(2x+7)\\&=(3x+5)(3x-5)(2x+7)\end{aligned}$$

135. $$\begin{aligned}x^3+4x^2+3x+12&=(x^3+4x^2)+(3x+12)\\&=x^2(x+4)+3(x+4)\\&=(x^2+3)(x+4)\end{aligned}$$

136. $$\begin{aligned}2x^3-3x^2+2x-3&=(2x^3-3x^2)+(2x-3)\\&=x^2(2x-3)+(2x-3)\\&=(x^2+1)(2x-3)\end{aligned}$$

3.2 Polynomial Functions

3.2 Practice Problems

1. a. $f(x)=\dfrac{x^2+1}{x-1}$ is not a polynomial function because its domain is not $(-\infty,\infty)$.

b. $g(x)=2x^7+5x^2-17$ is a polynomial function. Its degree is 7, the leading term is $2x^7$, and the leading coefficient is 2.

2. $$\begin{aligned}P(x)&=4x^3+2x^2+5x-17\\&=x^3\left[4+\frac{2}{x}+\frac{5}{x^2}-\frac{17}{x^3}\right]\end{aligned}$$

When $|x|$ is large, the terms $\frac{2}{x}$, $\frac{5}{x^2}$, and $-\frac{17}{x^3}$ are close to 0. Therefore, $P(x)=x^3(4+0+0-0)\approx 4x^3$.

3. Use the leading-term test to determine the end behavior of $y=f(x)=-2x^4+5x^2+3$. Here $n=4$ and $a_n=-2<0$. Thus, Case 2 applies. The end behavior is described as $y\to-\infty$ as $x\to-\infty$ and $y\to-\infty$ as $x\to\infty$.

4. First group the terms, then factor and solve $f(x)=0$:

$$\begin{aligned}f(x)&=2x^3-3x^2+4x-6\\&=2x^3+4x-3x^2-6\\&=2x(x^2+2)-3(x^2+2)\\&=(2x-3)(x^2+2)\end{aligned}$$

$0=(2x-3)(x^2+2)$

$0=2x-3\Rightarrow x=\frac{3}{2}$ or

$0=x^2+2$ (no real solution)

The only real zero is $\frac{3}{2}$.

5. $f(x)=2x^3-3x-6$

$f(1)=-7$ and $f(2)=4$. Since $f(1)$ and $f(2)$ have opposite signs, by the Intermediate Value Theorem, *f* has a real zero between 1 and 2.

6. $f(x)=(x+1)^2(x-3)(x+5)=0\Rightarrow$

$(x+1)^2=0$ or $x-3=0$ or $x+5=0\Rightarrow$

$x=-1$ or $x=3$ or $x=-5$

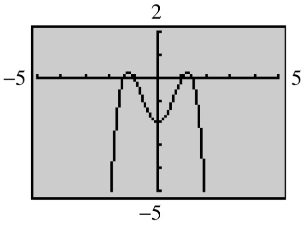
$f(x)$ has three distinct zeros.

7. $f(x)=(x-1)^2(x+3)(x+5)=0\Rightarrow$

$(x-1)^2=0$ or $x+3=0$ or $x+5=0\Rightarrow$

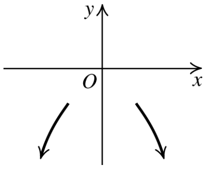
$x=1$ (multiplicity 2) or $x=-3$ (multiplicity 1) or $x=-5$ (multiplicity 1)

8. $f(x)=-x^4+3x^2-2$ has at most three turning points. Using a graphing calculator, we see that there are indeed, three turning points.



9. $f(x) = -x^4 + 5x^2 - 4$

Since the degree, 4, is even and the leading coefficient is −1, the end behavior is as shown:



$y \to -\infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.

Now find the zeros of the function:

$$0 = -x^4 + 5x^2 - 4 \Rightarrow 0 = -\left(x^4 - 5x^2 + 4\right)$$
$$= -\left(x^2 - 4\right)\left(x^2 - 1\right) \Rightarrow$$
$$0 = x^2 - 4 \text{ or } 0 = x^2 - 1 \Rightarrow$$
$$x = \pm 2 \quad \text{or } x = \pm 1$$

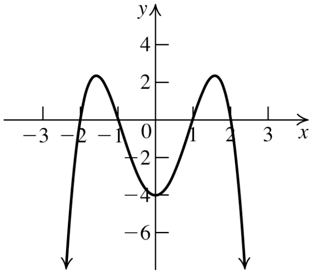
There are four zeros, each of multiplicity 1, so the graph crosses the *x*-axis at each zero. Next, find the *y*-intercept:

$$f(0) = -x^4 + 5x^2 - 4 = -4$$

Now find the intervals on which the graph lies above or below the *x*-axis. The four zeros divide the *x*-axis into five intervals, $(-\infty, -2)$, $(-2, -1), (-1, 1), (1, 2)$, and $(2, \infty)$. Determine the sign of a test value in each interval

Interval	Test point	Value of $f(x)$	Above/below *x*-axis
$(-\infty, -2)$	−3	−40	below
$(-2, -1)$	−1.5	2.1875	above
$(-1, 1)$	0	−4	below
$(1, 2)$	1.5	2.1875	above
$(2, \infty)$	3	−40	below

Plot the zeros, *y*-intercepts, and test points, and then join the points with a smooth curve.



10. $V = \dfrac{\pi}{3\sqrt{3}} x^3 = \dfrac{\pi}{3\sqrt{3}} \cdot 7^3 \approx 207.378 \text{ dm}^3$
≈ 207.378 L

3.2 Basic Concepts and Skills

1. The degree of $2x^5 - 3x^4 + x - 6$ is <u>5</u>, its leading term is $\underline{2x^5}$, its leading coefficient is <u>2</u>, and its constant term is <u>−6</u>.
2. The domain of a polynomial function is <u>the set of all real numbers.</u>
3. The behavior of the function $y = f(x)$ as $x \to \infty$ or $x \to -\infty$ is called the <u>end behavior</u> of the function.
4. The end behavior of any polynomial function is determined by its <u>leading</u> term.
5. A number *c* for which $f(c) = 0$ is called a <u>zero</u> of the function *f*.
6. If *c* is a zero of even multiplicity for a function *f*, then the graph of *f* <u>touches</u> the *x*-axis at *c*.
7. The graph of a polynomial function of degree *n* has at most <u>*n* – 1</u> turning points.
8. The zeros of a polynomial function *y* = *f*(*x*) are determined by solving the equation <u>*f*(*x*) = 0</u>.
9. Polynomial function; degree: 5; leading term: $2x^5$; leading coefficient: 2
10. Polynomial function; degree: 4; leading term: $-7x^4$; leading coefficient: –7
11. Polynomial function; degree: 3; leading term: $\frac{2}{3}x^3$; leading coefficient: $\frac{2}{3}$
12. Polynomial function; degree: 3; leading term: $\sqrt{2}x^3$; leading coefficient: $\sqrt{2}$
13. Polynomial function; degree: 4; leading term: πx^4; leading coefficient: π
14. Polynomial function; degree: 0; leading term: 5; leading coefficient: 5
15. Not a polynomial function: the graph has sharp corners; not a smooth curve; presence of $|x|$
16. Not a polynomial function: the domain is restricted – not $(-\infty, \infty)$

17. Not a polynomial function: the domain is not $(-\infty, \infty)$

18. Not a polynomial function: the graph is not continuous

19. Not a polynomial function: presence of $\sqrt{x}$

20. Not a polynomial function: noninteger exponent

21. Not a polynomial function: the domain is not $(-\infty, \infty)$

22. Not a polynomial function: negative exponent

23. Not a polynomial function: graph is not continuous

24. Not a polynomial function: the domain is restricted – not $(-\infty, \infty)$

25. Not a polynomial function: the graph is not continuous

26. Not a polynomial function: the graph has sharp corners; not a smooth curve

27. Not a polynomial function: not the graph of a function

28. Not a polynomial function: the graph has sharp corners; not a smooth curve

29. c **30.** f **31.** a

32. e **33.** d **34.** b

35. a. Zeros: $x = –5, 1$

b. $x = –5$: multiplicity: 1, crosses the x-axis;
$x = 1$, multiplicity: 1, crosses the x-axis

c. Maximum number of turning points: 1

36. a. Zeros: $x = –2, 3$

b. $x = –2$: multiplicity: 2, touches but does not cross the x-axis;
$x = 3$, multiplicity: 1, crosses the x-axis

c. Maximum number of turning points: 2

37. a. Zeros: $x = –1, 1$

b. $x = –1$: multiplicity: 2, touches but does not cross the x-axis;
$x = 1$, multiplicity: 3, crosses the x-axis

c. Maximum number of turning points: 4

38. a. Zeros: $x = –3, –1, 6$

b. $x = –3$: multiplicity: 2, touches but does not cross the x-axis;
$x = –1$, multiplicity: 1, crosses the x-axis
$x = 6$, multiplicity: 1, crosses the x-axis

c. Maximum number of turning points: 3

39. a. Zeros: $x = -\frac{2}{3}, \frac{1}{2}$

b. $x = -2/3$: multiplicity: 1, crosses the x-axis;
$x = 1/2$: multiplicity: 2, touches but does not cross the x-axis

c. Maximum number of turning points: 2

40. a. $f(x) = x^3 - 6x^2 + 8x = x(x^2 - 6x + 8)$
$= x(x-4)(x-2)$
Zeros: $x = 0, 2, 4$

b. $x = 0$: multiplicity: 1, crosses the x-axis;
$x = 2$, multiplicity: 1, crosses the x-axis;
$x = 4$, multiplicity: 1, crosses the x-axis

c. Maximum number of turning points: 2

41. a. $f(x) = -x^4 + 6x^3 - 9x^2 = -x^2(x^2 - 6x + 9)$
$= -x^2(x-3)^2$
Zeros: $x = 0, 3$

b. $x = 0$: multiplicity: 2, touches but does not cross the x-axis;
$x = 3$, multiplicity: 2, touches but does not cross the x-axis

c. Maximum number of turning points: 3

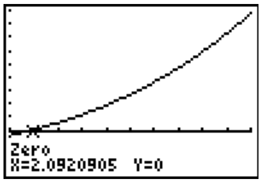
42. a. $f(x) = x^4 - 5x^2 - 36 = (x^2 - 9)(x^2 + 4)$
$= (x-3)(x+3)(x^2 + 4)$
Zeros: $x = –3, 3$

b. $x = –3$: multiplicity: 1, crosses the x-axis;
$x = 3$, multiplicity: 1, crosses the x-axis

c. Maximum number of turning points: 3

43. $f(2) = 2^4 - 2^3 - 10 = -2;$
$f(3) = 3^4 - 3^3 - 10 = 44$.

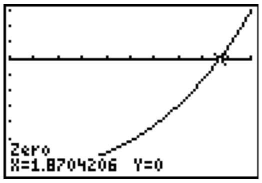
Because the sign changes, there is a real zero between 2 and 3. The zero is approximately 2.09.



44. $f(1) = 1^4 - 1^2 - 2(1) - 5 = -7;$

$f(2) = 2^4 - 2^2 - 2(2) - 5 = 3.$

Because the sign changes, there is a real zero between 1 and 2. The zero is approximately 1.87.

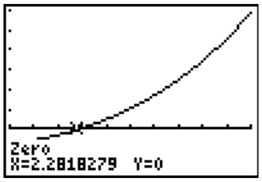


Zero
X=1.8704206 Y=0

45. $f(2) = 2^5 - 9(2)^2 - 15 = -19;$

$f(3) = 3^5 - 9(3)^2 - 15 = 147.$

Because the sign changes, there is a real zero between 2 and 3. The zero is approximately 2.28.

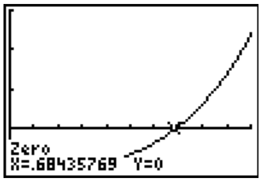


Zero
X=2.2818279 Y=0

46. $f(0) = 0^5 + 5(0^4) + 8(0^3) + 4(0^2) - 0 - 5 = -5;$

$f(1) = 1^5 + 5(1^4) + 8(1^3) + 4(1^2) - 1 - 5 = 12.$

Because the sign changes, there is a real zero between 0 and 1. The zero is approximately 0.68.



Zero
X=.68435769 Y=0

47. a. $y \to \infty$ as $x \to -\infty$; $y \to \infty$ as $x \to \infty$

b. $x^2 + 3 = 0 \Rightarrow x^2 = -3 \Rightarrow x = \pm i\sqrt{3} \Rightarrow$ there are no real zeros.

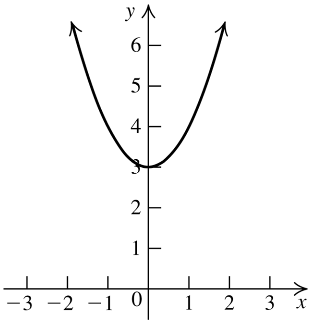
c. The graph is above the x-axis on $(-\infty, \infty)$.

d. $y = 0^2 + 3 = 3 \Rightarrow$ the y-intercept is 3.

e. $f(-x) = (-x)^2 + 3 = x^2 + 3 \Rightarrow f$ is even.
The graph is symmetrical with respect to the y-axis.

f. Maximum number of turning points: 1

g.



48. a. $y \to \infty$ as $x \to -\infty$; $y \to \infty$ as $x \to \infty$

b. $x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x = 2.$
The graph touches but does not cross the x-axis.

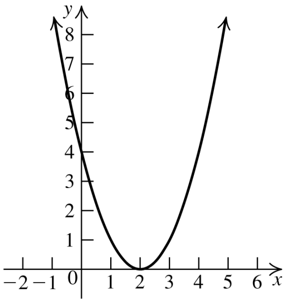
c. The graph is above the x-axis on $(-\infty, 2) \cup (2, \infty)$.

d. $y = 0^2 - 4(0) + 4 = 4 \Rightarrow$ the y-intercept is 4.

e. $f(-x) = (-x)^2 - 4(-x) + 4 = x^2 + 4x + 4$
$\neq f(x) \Rightarrow f$ is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries with respect to the axes or the origin. $h = 2$, so the graph is symmetric about $x = 2$.

f. Maximum number of turning points: 1

g.



49. a. $y \to \infty$ as $x \to -\infty$; $y \to \infty$ as $x \to \infty$

b. $x^2 + 4x - 21 = 0 \Rightarrow (x+7)(x-3) = 0 \Rightarrow$
$x = 3$ or $x = -7$.
The graph crosses the x-axis at both zeros.

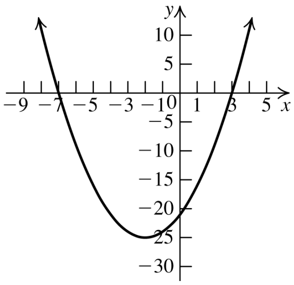
c. The graph is above the x-axis on $(-\infty, -7) \cup (3, \infty)$ and below the x-axis on $(-7, 3)$.

d. $y = 0^2 + 4(0) - 21 = -21 \Rightarrow$ the y-intercept is -21.

e. $f(-x) = (-x)^2 + 4(-x) - 21$
$= x^2 - 4x - 21 \neq f(x) \Rightarrow f$ is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries with respect to the axes or the origin.
$h = -\dfrac{b}{2a} = -\dfrac{4}{2(1)} = -2,$ so the graph is symmetric about $x = -2$.

f. Maximum number of turning points: 1

g.



50. a. $y \to -\infty$ as $x \to -\infty$; $y \to -\infty$ as $x \to \infty$

b. $-x^2 + 4x + 12 = 0 \Rightarrow -(x^2 - 4x - 12) = 0 \Rightarrow$ $-(x-6)(x+2) = 0 \Rightarrow x = 6$ or $x = -2$.
The graph crosses the x-axis at both zeros.

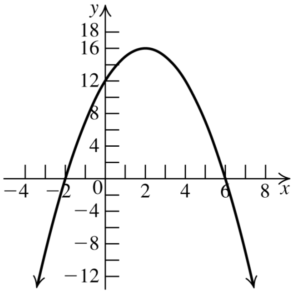
c. The graph is above the x-axis on $(-2, 6)$ and below the x-axis on $(-\infty, -2) \cup (6, \infty)$.

d. $y = -0^2 + 4(0) + 12 = 12 \Rightarrow$ the y-intercept is 12.

e. $f(-x) = -(-x)^2 + 4(-x) + 12$
$= -x^2 - 4x + 12 \neq f(x) \Rightarrow f$ is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries with respect to the axes or the origin.
$h = -\dfrac{b}{2a} = -\dfrac{4}{2(-1)} = 2,$ so the graph is symmetric about $x = 2$.

f. Maximum number of turning points: 1

g.



51. a. $y \to \infty$ as $x \to -\infty$; $y \to -\infty$ as $x \to \infty$

b. $-2x^2(x+1) = 0 \Rightarrow x = 0 \cup x = -1.$
The graph touches but does not cross the x-axis at $x = 0$. The graph crosses the x-axis at $x = -1$.

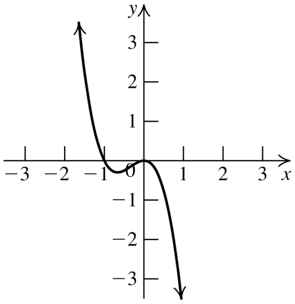
c. The graph is above the x-axis on $(-\infty, -1)$ and below the x-axis on $(-1, 0) \cup (0, \infty)$.

d. $y = -2(0)^2(0+1) = 0 \Rightarrow$ the y-intercept is 0.

e. $f(-x) = 2(-x)^2(-x+1) = 2x^2(-x+1)$
$\neq f(x) \Rightarrow f$ is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 2

g.



52. a. $y \to \infty$ as $x \to -\infty$; $y \to -\infty$ as $x \to \infty$

b. $x - x^3 = 0 \Rightarrow x(1 - x^2) = 0 \Rightarrow$
$x(1-x)(1+x) = 0 \Rightarrow x = 0$ or $x = 1$ or $x = -1$.
The graph crosses the x-axis at all three zeros.

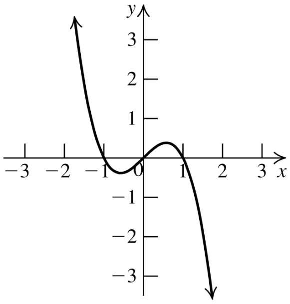
c. The graph is above the x-axis on $(-\infty, -1)$ and $(0, 1)$ and below the x-axis on $(-1, 0)$ and $(1, \infty)$.

d. $y = 0 - 0^3 = 0 \Rightarrow$ the y-intercept is 0.

e. $f(-x) = -x - (-x)^3 = -x + x^3 = -f(x) \Rightarrow$ f is odd, so f is symmetrical with respect to the origin.

f. Maximum number of turning points: 2

g.



53. a. $y \to \infty$ as $x \to -\infty$; $y \to \infty$ as $x \to \infty$

b. $x^2(x-1)^2 = 0 \Rightarrow x = 0$ or $x = 1$.
The graph touches but does not cross the x-axis at both zeros.

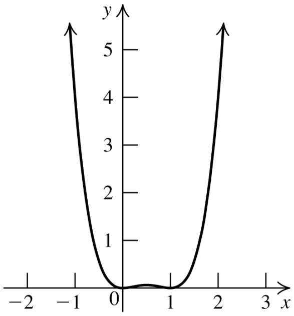
c. The graph is above the x-axis on $(-\infty, 0) \cup (0,1) \cup (1, \infty)$.

d. $y = 0^2(0-1)^2 = 0 \Rightarrow$ the y-intercept is 0.

e. $f(-x) = (-x)^2(-x-1)^2 = x^2(x^2+2x+1)$
$\neq f(x) \Rightarrow$ f is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 3

g.



54. a. $y \to \infty$ as $x \to -\infty$; $y \to \infty$ as $x \to \infty$

b. $x^2(x+1)(x-2) = 0 \Rightarrow x = 0$ or $x = -1$ or $x = 2$. The graph touches but does not cross the x-axis at $x = 0$. The graph crosses the x-axis at $x = -1$ and $x = 2$.

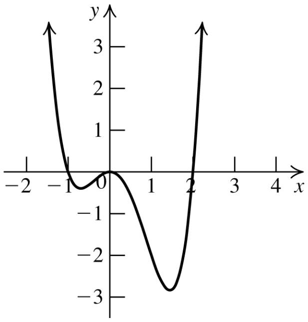
c. The graph is above the x-axis on $(-\infty, -1)$ and $(2, \infty)$ and below the x-axis on $(-1, 0) \cup (0, 2)$.

d. $y = 0^2(0+1)(0-2) = 0 \Rightarrow$ the y-intercept is 0.

e. $f(-x) = (-x)^2(-x+1)(-x-2)$
$= x^2(-x+1)(-x-2) = x^2(x^2+x-2)$
$\neq f(x) \Rightarrow$ f is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 3

g.



55. a. $y \to \infty$ as $x \to -\infty$; $y \to \infty$ as $x \to \infty$

b. $(x-1)^2(x+3)(x-4) = 0 \Rightarrow x = 1$ or $x = -3$ or $x = 4$. The graph touches but does not cross the x-axis at $x = 1$. The graph crosses the x-axis at $x = -3$ and $x = 4$.

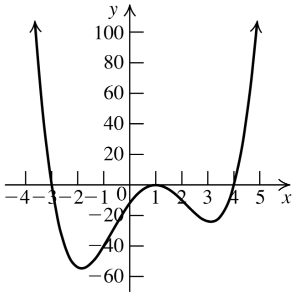
c. The graph is above the x-axis on $(-\infty, -3)$ and $(4, \infty)$ and below the x-axis on $(-3, 1) \cup (1, 4)$.

d. $y = (0-1)^2(0+3)(0-4) = -12 \Rightarrow$ the y-intercept is –12.

e.
$$\begin{aligned} f(x) &= (x-1)^2(x+3)(x-4) \\ &= (x^2-2x+1)(x^2-x-12) \\ &= x^4-3x^3-9x^2+23x-12. \end{aligned}$$
$$\begin{aligned} f(-x) &= (-x-1)^2(-x+3)(-x-4) \\ &= (x^2+2x+1)(x^2+x-12) \\ &= x^4+3x^3-9x^2-23x-12. \end{aligned}$$
$f(x) \neq f(x) \Rightarrow$ f is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 3

g.



56. a. $y \to \infty$ as $x \to -\infty$; $y \to \infty$ as $x \to \infty$

b. $(x+1)^2(x^2+1) = 0 \Rightarrow x = -1$. The graph touches but does not cross the x-axis at $x = -1$.

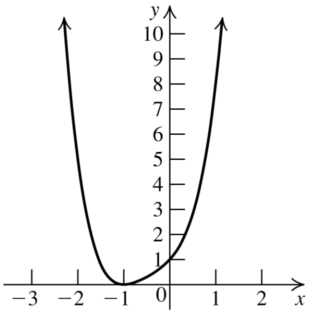
c. The graph is above the x-axis on $(-\infty,-1)\bigcup(-1,\infty)$.

d. $y=(0+1)^2(0^2+1)=1\Rightarrow$ the y-intercept is 1.

e.
$$\begin{aligned}f(x)&=(x+1)^2(x^2+1)\\&=(x^2+2x+1)(x^2+1)\\&=x^4+2x^3+2x^2+2x+1.\end{aligned}$$
$$\begin{aligned}f(-x)&=(-x+1)^2((-x)^2+1)\\&=(x^2-2x+1)(x^2+1)\\&=x^4-2x^3+2x^2-2x+1.\end{aligned}$$
$f(x)\neq f(x)\Rightarrow$ f is not even.
$f(-x)\neq -f(x)\Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 1

g.



57. a. $y\to\infty$ as $x\to-\infty$; $y\to-\infty$ as $x\to\infty$

b. $-x^2(x-1)^2(x+1)=0\Rightarrow x=0$ or $x=1$ or $x=-1$. The graph touches but does not cross the x-axis at $x=0$ and $x=-1$. The graph crosses the x-axis at $x=1$.

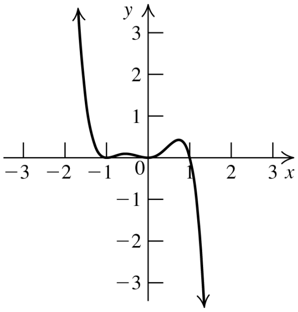
c. The graph is above the x-axis on $(-\infty,-1)\cup(-1,0)\cup(0,1)$ and below the x-axis on $(1,\infty)$.

d. $y=-0^2(0-1)^2(0+1)=0\Rightarrow$ the y-intercept is 0.

e.
$$\begin{aligned}f(x)&=-x^2(x^2-1)(x+1)\\&=-x^5-x^4+x^3+x^2.\end{aligned}$$
$$\begin{aligned}f(-x)&=-(-x)^2((-x)^2-1)(-x+1)\\&=x^5-x^4-x^3+x^2.\end{aligned}$$
$f(x)\neq f(x)\Rightarrow$ f is not even.
$f(-x)\neq -f(x)\Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 4

g.



58. a. $y\to\infty$ as $x\to-\infty$; $y\to-\infty$ as $x\to\infty$

b. $-x^2(x^2-4)(x+2)=0\Rightarrow$
$-x^2(x-2)(x+2)(x+2)=0\Rightarrow x=0$ or $x=2$ or $x=-2$.
The graph touches but does not cross the x-axis at $x=0$ and $x=-2$. The graph crosses the x-axis at $x=2$.

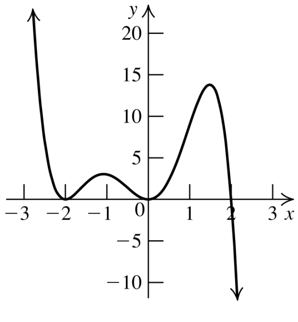
c. The graph is above the x-axis on $(-\infty,-2)\bigcup(-2,0)\bigcup(0,2)$ and below the x-axis on $(2,\infty)$.

d. $y=-0^2(0^2-4)(0+2)=0\Rightarrow$ the y-intercept is 0.

e.
$$\begin{aligned}f(x)&=-x^2(x^2-4)(x+2)\\&=-x^5-2x^4+4x^3+8x^2.\end{aligned}$$
$$\begin{aligned}f(-x)&=-(-x)^2((-x)^2-4)(-x+2)\\&=x^5-2x^4-4x^3+8x^2.\end{aligned}$$
$f(x)\neq f(x)\Rightarrow$ f is not even.
$f(-x)\neq -f(x)\Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 4

g.



59. a. $y\to\infty$ as $x\to-\infty$; $y\to\infty$ as $x\to\infty$

b. $x(x+1)(x-1)(x+2)=0\Rightarrow x=0$ or $x=-1$ or $x=-2$ or $x=1$. The graph crosses the x-axis at all four zeros.

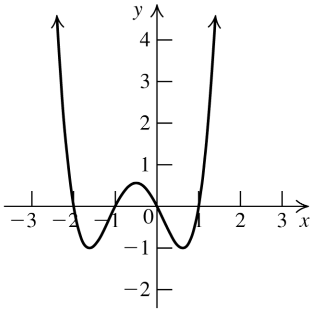
c. The graph is above the x-axis on $(-\infty,-2),\bigcup(-1,0)\bigcup(1,\infty)$. The graph is below the x-axis on $(-2,-1)\bigcup(0,1)$.

d. $y = 0(0+1)(0-1)(0+2) = 0 \Rightarrow$ the y-intercept is 0.

e. $$\begin{aligned} f(x) &= x(x+1)(x-1)(x+2) \\ &= x^4 + 2x^3 - x^2 - 2x. \\ f(-x) &= -x(-x+1)(-x-1)(-x+2) \\ &= x^4 - 2x^3 - x^2 + 2x. \end{aligned}$$
$f(x) \neq f(x) \Rightarrow f$ is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 3

g.



60. a. $y \to -\infty$ as $x \to -\infty$; $y \to \infty$ as $x \to \infty$

b. $x^2\left(x^2+1\right)\left(x-2\right) = 0 \Rightarrow x = 0$ or $x = 2$.
The graph touches but does not cross the x-axis at $x = 0$. The graph crosses the x-axis at $x = 2$.

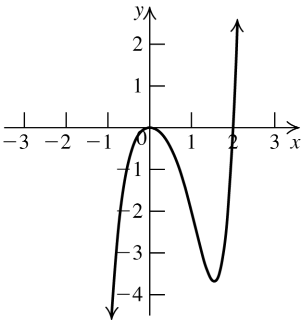
c. The graph is above the x-axis on $(2,\infty)$. The graph is below the x-axis on $(-\infty,0)\bigcup(0,2)$.

d. $y = 0^2(0+1)^2(0-2) = 0 \Rightarrow$ the y-intercept is 0.

e. $$\begin{aligned} f(x) &= x^2(x+1)^2(x-2) \\ &= x^5 - 3x^3 - 2x^2. \\ f(-x) &= (-x)^2(-x+1)^2(-x-2) \\ &= -x^5 + 3x^3 - 2x^2. \end{aligned}$$
$f(x) \neq f(x) \Rightarrow f$ is not even.
$f(-x) \neq -f(x) \Rightarrow f$ is not odd.
There are no symmetries.

f. Maximum number of turning points: 2

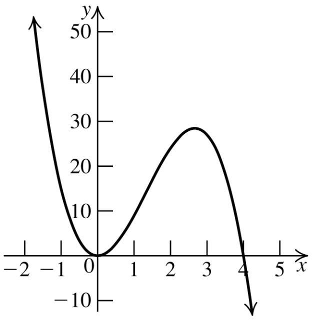
g.



3.2 Applying the Concepts

61. a. $3x^2(4-x) = 0 \Rightarrow x = 0$ or $x = 4$. $x = 0$, multiplicity 2; $x = 4$, multiplicity 1.

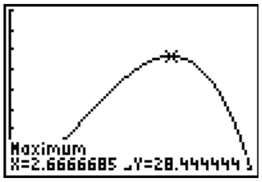
b.



c. There are 2 turning points.

d. Domain: [0, 4]. The portion between the x-intercepts is the graph of $R(x)$.

62.

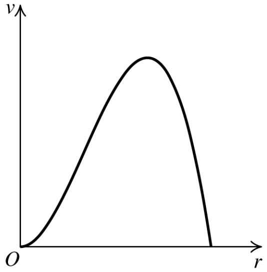


Maximum
X=2.6666685 Y=28.444444

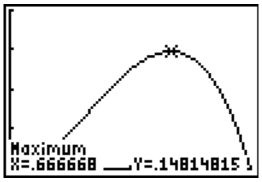
$R(x)$ is a maximum at $x \approx 2.67$.

63. a. Domain [0, 1]

b.



64.



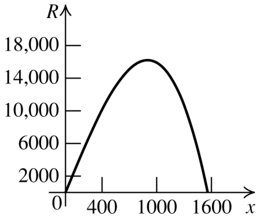
```
Maximum
X=.666668     Y=.14814815
```

v is a maximum when $r \approx 0.67$.

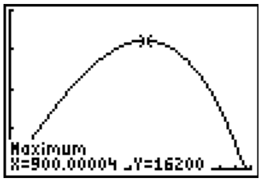
65. a. $R(x) = x\left(27 - \left(\dfrac{x}{300}\right)^2\right) = 27x - \dfrac{x^3}{90,000}$

b. The domain of $R(x)$ is the same as the domain of p. $p \ge 0$ when $p \le 900\sqrt{3}$. The domain is $\left[0,\ 900\sqrt{3}\right]$.

c.



66. a.



```
Maximum
X=900.00004  Y=16200
```

About 900 pairs of slacks were sold.

b. \$16,200

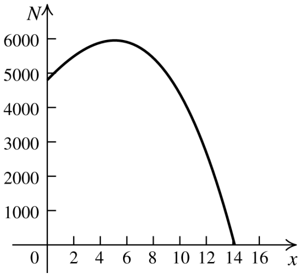
67. a. $N(x) = (x+12)\left(400 - 2x^2\right)$

b. The low end of the domain is 0. (There cannot be fewer than 0 workers.) The upper end of the domain is the value where productivity is 0, so solve $N(x) = 0$ to find the upper end of the domain.

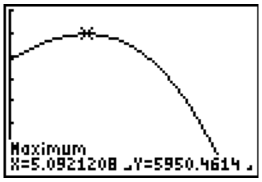
$(x+12)(400-2x^2) = 0 \Rightarrow x = -12$ (reject this) or $400 - 2x^2 = 0 \Rightarrow 200 = x^2 \Rightarrow$ $x = \pm 10\sqrt{2}$ (reject the negative solution).

The domain is $[0, 10\sqrt{2}]$.

c.



68. a.

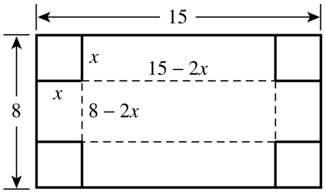


```
Maximum
X=5.0921208  Y=5950.4614
```

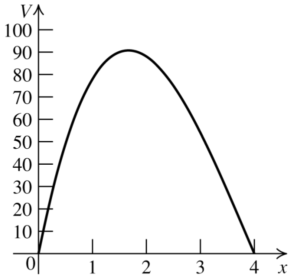
The maximum number of oranges that can be picked per hour is about 5950.

b. The number of employees = 12 + 5 = 17.

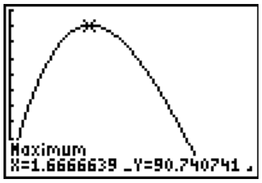
69. a. $V(x) = x(8-2x)(15-2x)$



b.



70.

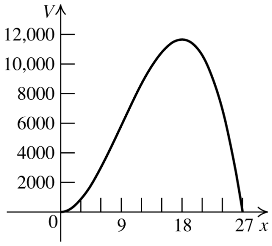


```
Maximum
X=1.6666639  Y=90.740741
```

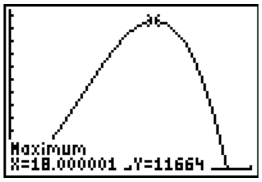
The largest possible value of the volume of the box is 90.74 cubic units.

71. a. $V(x) = x^2(108 - 4x)$

b.



72.

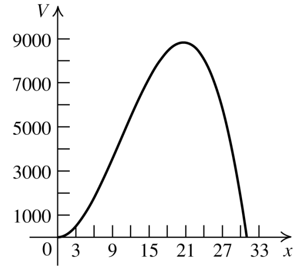


```
Maximum
X=18.000001  Y=11664
```

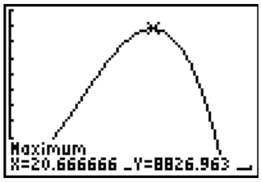
The largest possible volume of the box is 11,664 cubic inches.

73. a. $V(x) = x^2(62 - 2x)$

b.



74.

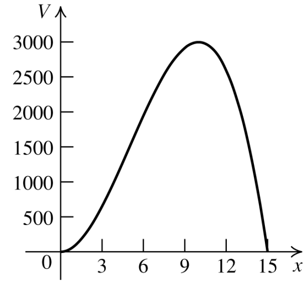


Maximum
X=20.666666 Y=8826.963

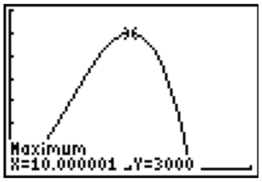
The volume is greatest when $x \approx 20.67$, so the dimensions of the suitcase with the largest volume are approximately
20.67 in. × 20.67 in. × 20.67 in.

75. a. $V(x) = 2x^2(45 - 3x) = 6x^2(15 - x)$

b.



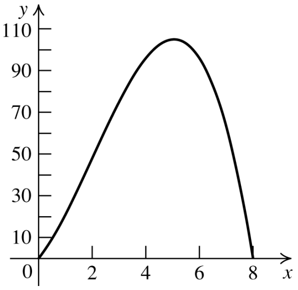
76.



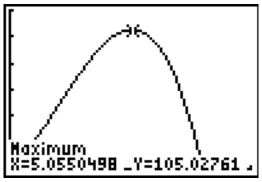
Maximum
X=10.000001 Y=3000

The volume is greatest when $x = 10$, so the width of the bag is 10 inches, the length is 2(10) = 20 inches, and the height is
45 – 20 – 10 = 15 inches.

77.



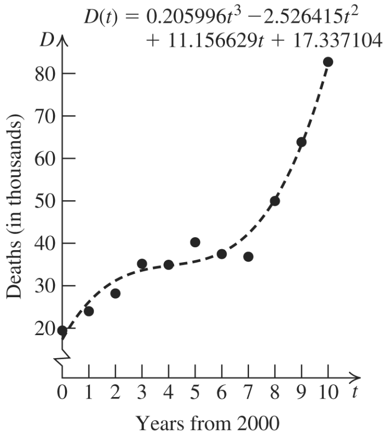
78.



Maximum
X=5.0550498 Y=105.02761

A worker is most efficient approximately five hours after 6:00 a.m. or 11:00 a.m.

79.



The model function $D(t)$ fits the actual data very well.

80. $D(11)$

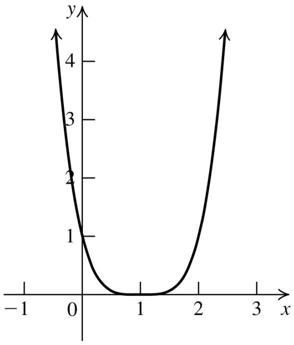
$$= 0.205996(11)^3 - 2.526415(11)^2 + 11.156629(11) + 17.337104$$
$$\approx 108.54 \text{ thousand} = 108,540$$

The number of deaths in the U.S. from FDA-approved drugs in 2011 was about 108,540. This is a valid estimate because the model function $D(t)$ can be used for near future estimates.

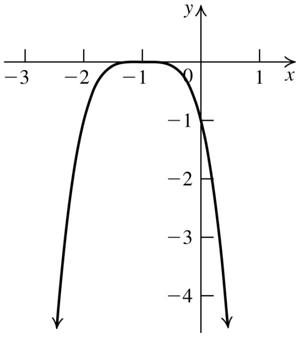
3.2 Beyond the Basics

81. The graph of $f(x) = (x-1)^4$ is the graph of $y = x^4$ shifted one unit right.

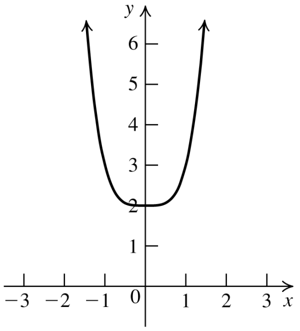
$(x-1)^4 = 0 \Rightarrow x = 1$. The zero is $x = 1$ with multiplicity 4.



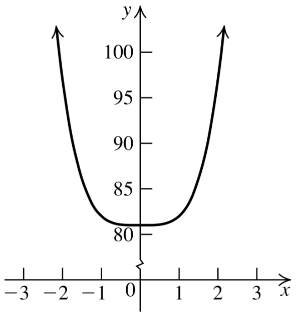
82. The graph of $f(x) = -(x+1)^4$ is the graph of $y = x^4$ shifted one unit left and then reflected about the *x*-axis. $-(x+1)^4 = 0 \Rightarrow x = -1$. The zero is $x = -1$ with multiplicity 4.



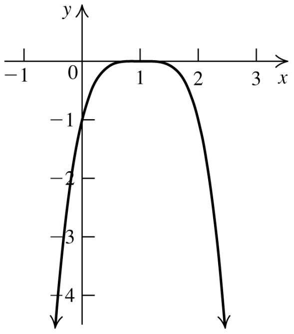
83. The graph of $f(x) = x^4 + 2$ is the graph of $y = x^4$ shifted two units up. $x^4 + 2 = 0 \Rightarrow x = \sqrt[4]{-2}$. There are no zeros.



84. The graph of $f(x) = x^4 + 81$ is the graph of $y = x^4$ shifted 81 units up. $x^4 + 81 = 0 \Rightarrow x = \sqrt[4]{-81}$. There are no zeros.



85. The graph of $f(x) = -(x-1)^4$ is the graph of $y = x^4$ shifted one unit right and then reflected about the *x*-axis. $-(x-1)^4 = 0 \Rightarrow x = 1$. The zero is $x = 1$ with multiplicity 4.

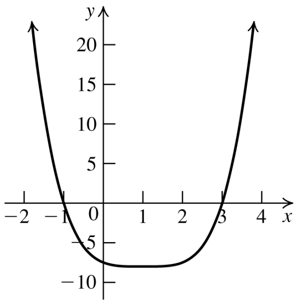


86. The graph of $f(x) = \frac{1}{2}(x-1)^4 - 8$ is the graph of $y = x^4$ shifted one unit right, compressed vertically by a factor of 2, and then shifted eight units down.

$\frac{1}{2}(x-1)^4 - 8 = 0 \Rightarrow$

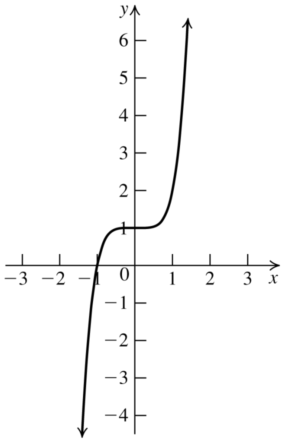
$(x-1)^4 = 16 \Rightarrow x - 1 = \pm 2 \Rightarrow x = 3$ or $x = -1$.

Both zeros have multiplicity 1.

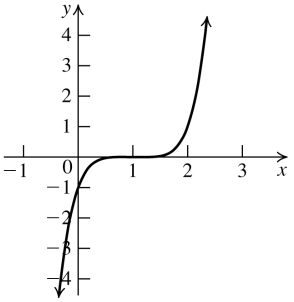


87. The graph of $f(x) = x^5 + 1$ is the graph of $y = x^5$ shifted one unit up. $x^5 + 1 = 0 \Rightarrow$ $x = \sqrt[5]{-1} \Rightarrow x = -1$.

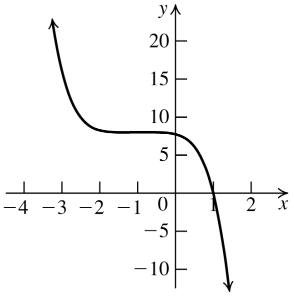
The zero is –1 with multiplicity 1.



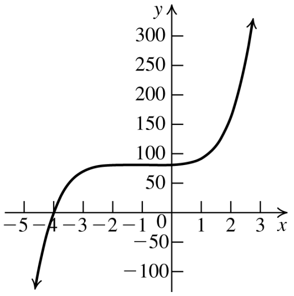
88. The graph of $f(x) = (x-1)^5$ is the graph of $y = x^5$ shifted one unit right. $(x-1)^5 = 0 \Rightarrow x = 1$. The zero is 1 with multiplicity 5.



89. The graph of $f(x) = 8 - \dfrac{(x+1)^5}{4}$ is the graph of $y = x^5$ shifted one unit left, compressed vertically by one-fourth, reflected about the *x*-axis, and then shifted up eight units.
$8 - \dfrac{(x+1)^5}{4} = 0 \Rightarrow -\dfrac{(x+1)^5}{4} = -8 \Rightarrow$
$(x+1)^5 = 32 \Rightarrow x+1 = 2 \Rightarrow x = 1.$
The zero is 1 with multiplicity 1.



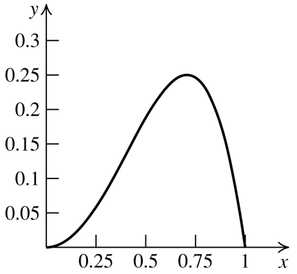
90. The graph of $f(x) = 81 + \dfrac{(x+1)^5}{3}$ is the graph of $y = x^5$ shifted one unit left, compressed vertically by one-third, and then shifted up 81 units. $81 + \dfrac{(x+1)^5}{3} = 0 \Rightarrow \dfrac{(x+1)^5}{3} = -81 \Rightarrow$ $(x+1)^5 = -243 \Rightarrow x+1 = -3 \Rightarrow x = -4.$ The zero is –4 with multiplicity 1.



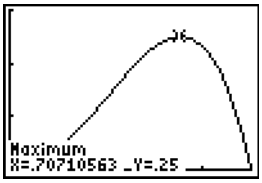
91. If $0 < x < 1$, then $x^2 < 1$. Multiplying both sides of the inequality by x^2, we obtain $x^4 < x^2$. Thus, $0 < x^4 < x^2$.

92. If $x > 1$, then $x^2 > 1$. Multiplying both sides of the inequality by x^2, we obtain $x^4 > x^2$. Thus, $x^4 > x^2 > 1$.

93.

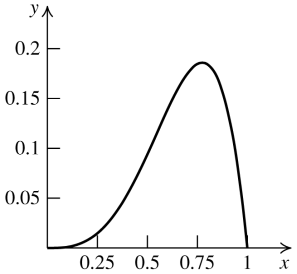


The graph of $y = x^2 - x^4$ represents the difference between the two functions $y = x^2$ and $y = x^4$, so the maximum distance between the graphs occurs at the local maximum of $y = x^2 - x^4$. The maximum vertical distance is 0.25. It occurs at $x \approx 0.71$.



Maximum
X=.70710563 Y=.25

94.



The graph of $y = x^3 - x^5$ represents the difference between the two functions $y = x^3$ and $y = x^5$, so the maximum distance between the graphs occurs at the local maximum of $y = x^3 - x^5$. The maximum vertical distance is 0.19. It occurs at $x \approx 0.77$.

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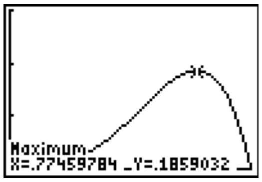

```
Maximum
X=.77459784  Y=.1859032
```

Answers will vary in exercises 95–98. Sample answers are given.

95. $f(x) = x^2(x+1)$

96. $f(x) = -x(x+1)(x-1)$

97. $f(x) = 1 - x^4$

98. $f(x) = x^2(x+1)(x-1)$

99. The smallest possible degree is 5, because the graph has five x-intercepts and four turning points.

100. The smallest possible degree is 5, because the graph has four turns.

101. The smallest possible degree is 6, because the graph has five turning points.

102. The smallest possible degree is 6, because the graph has five turning points.

3.2 Critical Thinking/Discussion/Writing

103. It is not possible for a polynomial function to have no y-intercepts because the domain of any polynomial function is $(-\infty, \infty)$, which includes the point $x = 0$.

104. It is possible for a polynomial function to have no x-intercepts because the function can be shifted above the x-axis. An example is the function $y = x^2 + 1$.

105. It is not possible for the graph of a polynomial function of degree 3 to have exactly one local maximum and no local minimum because the graph of a function of degree 3 rises in one direction and falls in the other. This requires an even number of turning points. Since the degree is 3, there can be only zero or two turning points. Therefore, if there is a local maximum, there must also be another turning point, which will be a local minimum.

106. It is not possible for the graph of a polynomial function of degree 4 to have exactly one local maximum and exactly one local minimum because the graph rises in both directions or falls in both directions. This requires an odd number of turning points. Since the degree is 4, there can be only one or three turning points. If there were exactly one local maximum and exactly one local minimum, there would be two turning points.

107. If $P(x) = a_0x^m + a_1x^{m-1} + \cdots a_{m-1}x + a_m$ and $Q(x) = b_0x^n + b_1x^{n-1} + \cdots b_{n-1}x + b_n$, then

$$\begin{aligned}(P \circ Q)(x) \\ &= a_0\left(b_0x^n + b_1x^{n-1} + \cdots b_{n-1}x + b_n\right)^m \\ &\quad + a_1\left(b_0x^n + b_1x^{n-1} + \cdots b_{n-1}x + b_n\right)^{m-1} + \cdots \\ &\quad + a_{m-1}\left(b_0x^n + b_1x^{n-1} + \cdots b_{n-1}x + b_n\right) + a_m,\end{aligned}$$

which is a polynomial of degree mn.

3.2 Maintaining Skills

108. $\dfrac{12x^5}{-3x^3} = -4x^{5-3} = -4x^2$

109. $\dfrac{x^5}{x^7} = x^{5-7} = x^{-2} = \dfrac{1}{x^2}$

110. $\dfrac{3x^4}{2x^2} = \dfrac{3x^{4-2}}{2} = \dfrac{3x^2}{2}$

111. $\dfrac{x^5 - 3x^2}{x^2} = \dfrac{x^2\left(x^3 - 3\right)}{x^2} = x^3 - 3$

112. $$\begin{aligned}\frac{5x^3 + 2x^2 - x}{x} &= \frac{x\left(5x^2 + 2x - 1\right)}{x} \\ &= 5x^2 + 2x - 1\end{aligned}$$

113. a. $7 = 1 \cdot 7$ **b.** $7 = 7 \cdot 1$

c. $7 = (-1)(-7)$ **d.** $7 = (-7)(-1)$

114. From exercise 113, the integer factors of 7 are 1, 7, –1, and –7.

115. The integer factors of 120 are –120, –60, –40, –30, –24, –20, –15, –12, –10, –8, –6, –5, –4, –3, –2, –1, 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120.

116. $f(x) = x^3 - 3x^2 + 2x - 9$
$f(-2) = (-2)^3 - 3(-2)^2 + 2(-2) - 9 = -33$

117. $g(x) = x^4 + 2x^3 - 20x - 5$
$g(3) = (3)^4 + 2(3)^3 - 20(3) - 5 = 70$

118. $g(x) = \left(-7x^5 + 2x\right)(x - 13)$

$g(13) = \left[-7(13)^5 + 2(13)\right]\left[(13) - 13\right] = 0$

119. $f(x) = \left(x^8 - 15x^7 + 5x^3\right)(x + 12)$

$f(-12)$

$= \left[(-12)^8 - 15(-12)^7 + 5(-12)^3\right]\left[(-12) + 12\right]$

$= 0$

120. $x^2 - 3x - 10 = (x - 5)(x + 2)$

121.
$$\begin{aligned} -2x^2 + x + 1 &= -\left(2x^2 - x - 1\right) \\ &= -(2x + 1)(x - 1) \\ &= (2x + 1)(1 - x) \end{aligned}$$

122. $(2x + 3)\left(x^2 + 6x - 7\right) = (2x + 3)(x + 7)(x - 1)$

123. $(x - 9)\left(6x^2 - 7x - 3\right) = (x - 9)(3x + 1)(2x - 3)$

3.3 Dividing Polynomials

3.3 Practice Problems

1.
$$\begin{array}{r} 3x + 4 \\ x^2 + 0x + 1 \overline{\big) 3x^3 + 4x^2 + x + 7} \\ (-)\underline{3x^3 + 0x^2 + 3x} \\ 4x^2 - 2x + 7 \\ (-)\ \underline{4x^2 + 0x + 4} \\ -2x + 3 \end{array}$$

Quotient: $3x + 4$; remainder: $-2x + 3$

2.
$$\begin{array}{r} x^2 + x + 3 \\ x^2 - x + 3 \overline{\big) x^4 + 0x^3 + 5x^2 + 2x + 6} \\ \underline{x^4 - x^3 + 3x^2} \\ x^3 + 2x^2 + 2x \\ \underline{x^3 - x^2 + 3x} \\ 3x^2 - x + 6 \\ \underline{3x^2 - 3x + 9} \\ 2x - 3 \end{array}$$

Quotient: $x^2 + x + 3$; remainder: $2x - 3$ or

$x^2 + x + 3 + \dfrac{2x - 3}{x^2 - x + 3}$

3.
$$\begin{array}{r|rrrr} \underline{3} & 2 & -7 & 0 & 5 \\ & & 6 & -3 & -9 \\ \hline & 2 & -1 & -3 & -4 \end{array}$$

The quotient is $2x^2 - x - 3$ remainder -4 or

$2x^2 - x - 3 - \dfrac{4}{x - 3}.$

4.
$$\begin{array}{r|rrrr} \underline{-3} & 2 & 1 & -18 & -7 \\ & & -6 & 15 & 9 \\ \hline & 2 & -5 & -3 & 2 \end{array}$$

The quotient is $2x^2 - 5x - 3$ remainder 2 or

$2x^2 - 5x - 3 + \dfrac{2}{x + 3}.$

5. $F(1) = 1^{110} - 2 \cdot 1^{57} + 5 = 1 - 2 + 5 = 4$, so the remainder when $F(x) = x^{110} - 2x^{57} + 5$ is divided by $x - 1$ is 4.

6.
$$\begin{array}{r|rrrrr} \underline{-2} & 1 & 0 & 10 & 2 & -20 \\ & & -2 & 4 & -28 & 52 \\ \hline & 1 & -2 & 14 & -26 & 32 \end{array}$$

The remainder is 32, so $f(-2) = 32$.

7. Since -2 is a zero of the function $3x^3 - x^2 - 20x - 12$, $x + 2$ is a factor. Use synthetic division to find the depressed equation.

$$\begin{array}{r|rrrr} \underline{-2} & 3 & -1 & -20 & -12 \\ & & -6 & 14 & 12 \\ \hline & 3 & -7 & -6 & 0 \end{array}$$

Thus,

$3x^3 - x^2 - 20x - 12 = (x + 2)\left(3x^2 - 7x - 6\right).$

Now solve $3x^2 - 7x - 6 = 0$ to find the two remaining zeros:

$3x^2 - 7x - 6 = 0 \Rightarrow (x - 3)(3x + 2) = 0 \Rightarrow$

$x - 3 = 0 \text{ or } 3x + 2 = 0 \Rightarrow x = 3 \text{ or } x = -\dfrac{2}{3}$

The solution set is $\left\{-2, -\dfrac{2}{3}, 3\right\}.$

8. $C(x) = 0.23x^3 - 4.255x^2 + 0.345x + 41.05$

$C(3) = 10 \Rightarrow C(x) = (x - 3)Q(x) + 10 \Rightarrow$

$C(x) - 10 = (x - 3)Q(x)$

So, 3 is zero of $C(x) - 10 =$

$0.23x^3 - 4.255x^2 + 0.345x + 31.05$.

We need to find another positive zero of $C(x) - 10$. Use synthetic division to find the depressed equation.

(*continued on next page*)

(*continued*)

$$\begin{array}{r|rrrr} \underline{3} & 0.23 & -4.255 & 0.345 & 31.05 \\ & & 0.69 & -10.695 & -31.05 \\ \hline & 0.23 & -3.565 & -10.35 & 0 \end{array}$$

Solve the depressed equation $0.23x^2-3.565x-10.35=0$ using the quadratic formula:

$$x=\frac{3.565\pm\sqrt{(-3.565)^2-4(0.23)(-10.35)}}{2(0.23)}$$
$$\approx 18 \text{ or } -2.5$$

The positive zero is 18.
Check by verifying that $C(18)=10$.

3.3 Basic Concepts and Skills

1. In the division $\dfrac{x^4-2x^3+5x^2-2x+1}{x^2-2x+3}=x^2+2+\dfrac{2x-5}{x^2-2x+3}$, the dividend is $\underline{x^4-2x^3+5x^2-2x+1}$, the divisor is $\underline{x^2-2x+3}$, the quotient is $\underline{x^2+2}$, and the remainder is $\underline{2x-5}$.

2. If $P(x), Q(x)$, and $F(x)$ are polynomials, and $F(x)=P(x)\cdot Q(x)$, then the factors of $F(x)$ are $\underline{P(x)}$ and $\underline{Q(x)}$.

3. The Remainder Theorem states that if a polynomial $F(x)$ is divided by $(x-a)$, then the remainder $R=\underline{F(a)}$.

4. The Factor Theorem states that $(x-a)$ is a factor of a polynomial $F(x)$ if and only if $\underline{F(a)}=0$.

5. True

6. False. See Practice Problem 1 for an example.

7.
$$\begin{array}{r} 3x-2 \\ 2x+1\overline{\big)6x^2-\ x-2} \\ \underline{-(6x^2+3x)} \\ -4x-2 \\ \underline{-(-4x-2)} \\ 0 \end{array}$$

8.
$$\begin{array}{r} 2x^2+2x+\frac{7}{2} \\ 2x-3\overline{\big)4x^3-2x^2+\ x-3} \\ \underline{-(4x^3-6x^2)} \\ 4x^2+\ x \\ \underline{-(4x^2-6x)} \\ 7x-3 \\ \underline{-\left(7x-\frac{21}{2}\right)} \\ \frac{15}{2} \end{array}$$

In exercises 9–14, insert zero coefficients for missing terms.

9.
$$\begin{array}{r} 3x^3-3x^2-3x+6 \\ x+1\overline{\big)3x^4+0x^3-6x^2+3x-7} \\ \underline{-(3x^4+3x^3)} \\ -3x^3-6x^2 \\ \underline{-(-3x^3-3x^2)} \\ -3x^2+3x \\ \underline{-(-3x^2-3x)} \\ 6x-7 \\ \underline{-(6x+6)} \\ -13 \end{array}$$

10.
$$\begin{array}{r} x^4-2x^2+5x+4 \\ x^2+0x+2\overline{\big)x^6+0x^5+0x^4+5x^3+0x^2+\ 7x+3} \\ \underline{-(x^6+0x^5+2x^4)} \\ -2x^4+5x^3+0x^2 \\ \underline{-(-2x^4+0x^3-4x^2)} \\ 5x^3+4x^2+\ 7x \\ \underline{-(5x^3+0x^2+10x)} \\ 4x^2-\ 3x+3 \\ \underline{-(4x^2+\ 0x+8)} \\ -3x-5 \end{array}$$

11.
$$\begin{array}{r} 2x-1 \\ 2x^2-x-5\overline{\big)4x^3-4x^2-\ 9x+5} \\ \underline{-(4x^3-2x^2-10x)} \\ -2x^2+\ x+5 \\ \underline{-(-2x^2+\ x+5)} \\ 0 \end{array}$$

12.
$$\begin{array}{r}
y^3+\ y^2+\ y-5 \\
y^2+2y-3\overline{)y^5+3y^4+0y^3-6y^2+2y-7} \\
\underline{-(y^5+2y^4-3y^3)\qquad\qquad\quad} \\
y^4+3y^3-6y^2\qquad\quad \\
\underline{-(y^4+2y^3-3y^2)\qquad\quad} \\
y^3-3y^2+\ 2y\quad \\
\underline{-(y^3+2y^2-\ 3y)\quad} \\
-5y^2+\ 5y-\ 7 \\
\underline{-(-5y^2-10y+15)} \\
15y-22
\end{array}$$

13.
$$\begin{array}{r}
z^2+2z+1 \\
z^2-2z+1\overline{)z^4+0z^3-2z^2+0z+1} \\
\underline{-(z^4-2z^3+\ z^2)\qquad\qquad} \\
2z^3-3z^2+0z\qquad \\
\underline{-(2z^3-4z^2+2z)\qquad} \\
z^2-2z+1 \\
\underline{-(z^2-2z+1)} \\
0
\end{array}$$

14. First, arrange the terms in order of descending powers.
$$\begin{array}{r}
2x^2-3x+2 \\
3x^2-x-5\overline{)6x^4-11x^3-\ x^2+13x-10} \\
\underline{-(6x^4-2x^3-10x^2)\qquad\qquad\quad} \\
-9x^3+\ 9x^2+13x\qquad \\
\underline{-(-9x^3+\ 3x^2+15x)\qquad} \\
6x^2-\ 2x-10 \\
\underline{-(6x^2-\ 2x-10)} \\
0
\end{array}$$

15.
$$\begin{array}{r|rrrr}
1 & 1 & -1 & -7 & 2 \\
 & & 1 & 0 & -7 \\ \hline
 & 1 & 0 & -7 & -5
\end{array}$$

The quotient is x^2-7 and the remainder is −5.

16.
$$\begin{array}{r|rrrr}
-2 & 2 & -3 & -1 & 2 \\
 & & -4 & 14 & -26 \\ \hline
 & 2 & -7 & 13 & -24
\end{array}$$

The quotient is $2x^2-7x+13$ and the remainder is −24.

17.
$$\begin{array}{r|rrrr}
-2 & 1 & 4 & -7 & -10 \\
 & & -2 & -4 & 22 \\ \hline
 & 1 & 2 & -11 & 12
\end{array}$$

The quotient is $x^2+2x-11$ and the remainder is 12.

18.
$$\begin{array}{r|rrrr}
3 & 1 & 1 & -13 & 2 \\
 & & 3 & 12 & -3 \\ \hline
 & 1 & 4 & -1 & -1
\end{array}$$

The quotient is x^2+4x-1 and the remainder is −1.

19.
$$\begin{array}{r|rrrrr}
2 & 1 & -3 & 2 & 4 & 5 \\
 & & 2 & -2 & 0 & 8 \\ \hline
 & 1 & -1 & 0 & 4 & 13
\end{array}$$

The quotient is x^3-x^2+4 and the remainder is 13.

20.
$$\begin{array}{r|rrrrr}
1 & 1 & -5 & -3 & 0 & 10 \\
 & & 1 & -4 & -7 & -7 \\ \hline
 & 1 & -4 & -7 & -7 & 3
\end{array}$$

The quotient is x^3-4x^2-7x-7 and the remainder is 3.

21.
$$\begin{array}{r|rrrr}
\frac{1}{2} & 2 & 4 & -3 & 1 \\
 & & 1 & \frac{5}{2} & -\frac{1}{4} \\ \hline
 & 2 & 5 & -\frac{1}{2} & \frac{3}{4}
\end{array}$$

The quotient is $2x^2+5x-\frac{1}{2}$ and the remainder is $\frac{3}{4}$.

22.
$$\begin{array}{r|rrrr}
\frac{1}{3} & 3 & 8 & 1 & 1 \\
 & & 1 & 3 & \frac{4}{3} \\ \hline
 & 3 & 9 & 4 & \frac{7}{3}
\end{array}$$

The quotient is $3x^2+9x+4$ and the remainder is $\frac{7}{3}$.

23.
$$\begin{array}{r|rrrr}
-\frac{1}{2} & 2 & -5 & 3 & 2 \\
 & & -1 & 3 & -3 \\ \hline
 & 2 & -6 & 6 & -1
\end{array}$$

The quotient is $2x^2-6x+6$ and the remainder is −1.

24. $$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & -2 & 8 & 2 \\ & & -1 & 1 & -3 \\ \hline & 3 & -3 & 9 & -1 \end{array}$$

The quotient is $3x^2 - 3x + 9$ and the remainder is −1.

25. $$\begin{array}{r|rrrrrr} 1 & 1 & 1 & -7 & 2 & 1 & -1 \\ & & 1 & 2 & -5 & -3 & -2 \\ \hline & 1 & 2 & -5 & -3 & -2 & -3 \end{array}$$

The quotient is $x^4 + 2x^3 - 5x^2 - 3x - 2$ and the remainder is −3.

26. $$\begin{array}{r|rrrrrr} -2 & 2 & 4 & -3 & -7 & 3 & -2 \\ & & -4 & 0 & 6 & 2 & -10 \\ \hline & 2 & 0 & -3 & -1 & 5 & -12 \end{array}$$

The quotient is $2x^4 - 3x^2 - x + 5$ and the remainder is −12.

27. $$\begin{array}{r|rrrrrr} -1 & 1 & 0 & 0 & 0 & 0 & 1 \\ & & -1 & 1 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & -1 & 1 & 0 \end{array}$$

The quotient is $x^4 - x^3 + x^2 - x + 1$ remainder 0.

28. $$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ & & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 2 \end{array}$$

The quotient is $x^4 + x^3 + x^2 + x + 1$ remainder 2.

29. $$\begin{array}{r|rrrrrrr} -1 & 1 & 0 & 2 & -1 & 0 & 0 & 5 \\ & & -1 & 1 & -3 & 4 & -4 & 4 \\ \hline & 1 & -1 & 3 & -4 & 4 & -4 & 9 \end{array}$$

The quotient is $x^5 - x^4 + 3x^3 - 4x^2 + 4x - 4$ remainder 9.

30. $$\begin{array}{r|rrrrrr} 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ & & 3 & 3 & 3 & 3 & 3 \\ \hline & 3 & 3 & 3 & 3 & 3 & 3 \end{array}$$

The quotient is $3x^4 + 3x^3 + 3x^2 + 3x + 3$ remainder 3.

31. a. $$\begin{array}{r|rrrr} 1 & 1 & 3 & 0 & 1 \\ & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & 5 \end{array}$$

The remainder is 5, so $f(1) = 5$.

b. $$\begin{array}{r|rrrr} -1 & 1 & 3 & 0 & 1 \\ & & -1 & -2 & 2 \\ \hline & 1 & 2 & -2 & 3 \end{array}$$

The remainder is 3, so $f(-1) = 3$.

c. $$\begin{array}{r|rrrr} \frac{1}{2} & 1 & 3 & 0 & 1 \\ & & \frac{1}{2} & \frac{7}{4} & \frac{7}{8} \\ \hline & 1 & \frac{7}{2} & \frac{7}{4} & \frac{15}{8} \end{array}$$

The remainder is $\frac{15}{8}$, so $f\left(\frac{1}{2}\right) = \frac{15}{8}$.

d. $$\begin{array}{r|rrrr} 10 & 1 & 3 & 0 & 1 \\ & & 10 & 130 & 1300 \\ \hline & 1 & 13 & 130 & 1301 \end{array}$$

The remainder is 1301, so $f(10) = 1301$.

32. a. $$\begin{array}{r|rrrr} -2 & 2 & -3 & 0 & 1 \\ & & -4 & 14 & -28 \\ \hline & 2 & -7 & 14 & -27 \end{array}$$

The remainder is –27, so $f(-2) = -27$.

b. $$\begin{array}{r|rrrr} -1 & 2 & -3 & 0 & 1 \\ & & -2 & 5 & -5 \\ \hline & 2 & -5 & 5 & -4 \end{array}$$

The remainder is –4, so $f(-1) = -4$.

c. $$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -3 & 0 & 1 \\ & & -1 & 2 & -1 \\ \hline & 2 & -4 & 2 & 0 \end{array}$$

The remainder is 0, so $f\left(-\frac{1}{2}\right) = 0$.

d. $$\begin{array}{r|rrrr} 7 & 2 & -3 & 0 & 1 \\ & & 14 & 77 & 539 \\ \hline & 2 & 11 & 77 & 540 \end{array}$$

The remainder is 540, so $f(7) = 540$.

33. a. $$\begin{array}{r|rrrrr} 1 & 1 & 5 & -3 & 0 & -20 \\ & & 1 & 6 & 3 & 3 \\ \hline & 1 & 6 & 3 & 3 & -17 \end{array}$$

The remainder is –17, so $f(1) = -17$.

b. $$\begin{array}{r|rrrrr} -1 & 1 & 5 & -3 & 0 & -20 \\ & & -1 & -4 & 7 & -7 \\ \hline & 1 & 4 & -7 & 7 & -27 \end{array}$$

The remainder is –27, so $f(-1) = -27$.

c. $$\begin{array}{r|rrrrr} -2 & 1 & 5 & -3 & 0 & -20 \\ & & -2 & -6 & 18 & -36 \\ \hline & 1 & 3 & -9 & 18 & -56 \end{array}$$

The remainder is –56, so $f(-2) = -56$.

d.
$$\begin{array}{r|rrrrr} 2 & 1 & 5 & -3 & 0 & -20 \\ & & 2 & 14 & 22 & 44 \\ \hline & 1 & 7 & 11 & 22 & 24 \end{array}$$
The remainder is 24, so $f(2) = 24$.

34. a.
$$\begin{array}{r|rrrrr} 0.1 & 1 & 0.5 & -0.3 & 0 & -20 \\ & & 0.1 & 0.06 & -0.024 & -0.0024 \\ \hline & 1 & 0.6 & -0.24 & -0.024 & -20.0024 \end{array}$$
The remainder is –20.0024, so $f(0.1) = -20.0024$.

b.
$$\begin{array}{r|rrrrr} 0.5 & 1 & 0.5 & -0.3 & 0 & -20 \\ & & 0.5 & 0.5 & 0.1 & 0.05 \\ \hline & 1 & 1.0 & 0.2 & 0.1 & -19.95 \end{array}$$
The remainder is –19.95, so $f(0.5) = -19.95$.

c.
$$\begin{array}{r|rrrrr} 1.7 & 1 & 0.5 & -0.3 & 0 & -20 \\ & & 1.7 & 3.74 & 5.848 & 9.9416 \\ \hline & 1 & 2.2 & 3.44 & 5.848 & -10.0584 \end{array}$$
The remainder is -10.0584, so $f(1.7) = -10.0584$.

d.
$$\begin{array}{r|rrrrr} -2.3 & 1 & 0.5 & -0.3 & 0 & -20 \\ & & -2.3 & 4.14 & -8.832 & 20.3136 \\ \hline & 1 & -1.8 & 3.84 & -8.832 & 0.3136 \end{array}$$
The remainder is 0.3136, so $f(-2.3) = 0.3136$.

35. $f(1) = 2(1)^3 + 3(1)^2 - 6(1) + 1 = 0 \Rightarrow x - 1$ is a factor of $2x^3 + 3x^2 - 6x + 1$.
Check as follows:
$$\begin{array}{r|rrrr} 1 & 2 & 3 & -6 & 1 \\ & & 2 & 5 & -1 \\ \hline & 2 & 5 & -1 & 0 \end{array}$$

36. $f(3) = 3(3)^3 - 9(3)^2 - 4(3) + 12 = 0 \Rightarrow x - 3$ is a factor of $3x^3 - 9x^2 - 4x + 12$.
Check as follows:
$$\begin{array}{r|rrrr} 3 & 3 & -9 & -4 & 12 \\ & & 9 & 0 & -12 \\ \hline & 3 & 0 & -4 & 0 \end{array}$$

37. $f(-1) = 5(-1)^4 + 8(-1)^3 + (-1)^2 + 2(-1) + 4 = 0 \Rightarrow x + 1$ is a factor of $5x^4 + 8x^3 + x^2 + 2x + 4$.
Check as follows:
$$\begin{array}{r|rrrrr} -1 & 5 & 8 & 1 & 2 & 4 \\ & & -5 & -3 & 2 & -4 \\ \hline & 5 & 3 & -2 & 4 & 0 \end{array}$$

38. $f(-3) = 3(-3)^4 + 9(-3)^3 - 4(-3)^2 - 9(-3) + 9 = 0 \Rightarrow x + 3$ is a factor of $3x^4 + 9x^3 - 4x^2 - 9x + 9$.
Check as follows:
$$\begin{array}{r|rrrrr} -3 & 3 & 9 & -4 & -9 & 9 \\ & & -9 & 0 & 12 & -9 \\ \hline & 3 & 0 & -4 & 3 & 0 \end{array}$$

39. $f(2) = 2^4 + 2^3 - 2^2 - 2 - 18 = 0 \Rightarrow x - 2$ is a factor of $x^4 + x^3 - x^2 - x - 18$.
Check as follows:
$$\begin{array}{r|rrrrr} 2 & 1 & 1 & -1 & -1 & -18 \\ & & 2 & 6 & 10 & 18 \\ \hline & 1 & 3 & 5 & 9 & 0 \end{array}$$

40. $f(-3) = (-3)^5 + 3(-3)^4 + (-3)^2 + 8(-3) + 15 = 0 \Rightarrow x + 3$ is a factor of $x^5 + 3x^4 + x^2 + 8x + 15$.
Check as follows:
$$\begin{array}{r|rrrrrr} -3 & 1 & 3 & 0 & 1 & 8 & 15 \\ & & -3 & 0 & 0 & -3 & -15 \\ \hline & 1 & 0 & 0 & 1 & 5 & 0 \end{array}$$

41. $f(-2) = (-2)^6 - (-2)^5 - 7(-2)^4 + (-2)^3 + 8(-2)^2 + 5(-2) + 2 = 0 \Rightarrow x + 2$ is a factor of
$x^6 - x^5 - 7x^4 + x^3 + 8x^2 + 5x + 2$.
Check as follows:
$$\begin{array}{r|rrrrrrr} -2 & 1 & -1 & -7 & 1 & 8 & 5 & 2 \\ & & -2 & 6 & 2 & -6 & -4 & -2 \\ \hline & 1 & -3 & -1 & 3 & 2 & 1 & 0 \end{array}$$

42. $f(2) = 2(2)^6 - 5(2)^5 + 4(2)^4 + (2)^3 - 7(2)^2 - 7(2) + 2 = 0 \Rightarrow x - 2$ is a factor of
$2x^6 - 5x^5 + 4x^4 + x^3 - 7x^2 - 7x + 2$.
Check as follows:
$$\begin{array}{r|rrrrrrr} 2 & 2 & -5 & 4 & 1 & -7 & -7 & 2 \\ & & 4 & -2 & 4 & 10 & 6 & -2 \\ \hline & 2 & -1 & 2 & 5 & 3 & -1 & 0 \end{array}$$

43. $f(-1) = 0 = (-1)^3 + 3(-1)^2 + (-1) + k \Rightarrow$
$0 = 1 + k \Rightarrow k = -1$

44. $f(1) = 0 = -1^3 + 4(1)^2 + k(1) - 2 \Rightarrow$
$1 + k = 0 \Rightarrow k = -1$

45. $f(2) = 0 = 2(2)^3 + (2^2)k - 2k - 2 \Rightarrow$
$14 + 2k = 0 \Rightarrow k = -7$

46. $f(1) = 0 = k^2 - 3(1^2)k - 2(1)k + 6 \Rightarrow$
$k^2 - 5k + 6 = 0 \Rightarrow (k-3)(k-2) = 0 \Rightarrow$
$k = 3$ or $k = 2$

In exercises 47–50, use synthetic division to find the remainder.

47.
$$\begin{array}{r|rrrr} 2 & -2 & 4 & -4 & 9 \\ & & -4 & 0 & -8 \\ \hline & -2 & 0 & -4 & 1 \end{array}$$

The remainder is 1, so $x - 2$ is not a factor of $-2x^3 + 4x^2 - 4x + 9.$

48.
$$\begin{array}{r|rrrr} -3 & -3 & -9 & 5 & 12 \\ & & 9 & 0 & -15 \\ \hline & -3 & 0 & 5 & -3 \end{array}$$

The remainder is –3, so $x + 3$ is not a factor of $-3x^3 - 9x^2 + 5x + 12.$

49.
$$\begin{array}{r|rrrrr} -2 & 4 & 9 & 3 & 1 & 4 \\ & & -8 & -2 & -2 & 2 \\ \hline & 4 & 1 & 1 & -1 & 6 \end{array}$$

The remainder is 6, so $x + 2$ is not a factor of $4x^4 + 9x^3 + 3x^2 + x + 4.$

50.
$$\begin{array}{r|rrrrr} 3 & 3 & -8 & 5 & 7 & -3 \\ & & 9 & 3 & 24 & 93 \\ \hline & 3 & 1 & 8 & 31 & 90 \end{array}$$

The remainder is 90, so $x - 3$ is not a factor of $3x^4 - 8x^3 + 5x^2 + 7x - 3.$

3.3 Applying the Concepts

51. $A = lw \Rightarrow l = \dfrac{A}{w} \Rightarrow$

$$\begin{array}{r} 2x^2+1 \\ x^2-x+2\overline{\big)2x^4-2x^3+5x^2-x+2} \\ -(\underline{2x^4-2x^3+4x^2}) \\ x^2-x+2 \\ -(\underline{x^2-x+2}) \\ 0 \end{array}$$

The width is $2x^2 + 1.$

52. $V = lwh \Rightarrow h = \dfrac{V}{lw}$
$lw = (x+3)(x+1) = x^2 + 4x + 3$

$$\begin{array}{r} x^2-x+1 \\ x^2+4x+3\overline{\big)x^4+3x^3+0x^2+\ x+3} \\ -(\underline{x^4+4x^3+3x^2}) \\ -x^3-3x^2+\ x \\ -(\underline{-x^3-4x^2-3x}) \\ x^2+4x+3 \\ -(\underline{x^2+4x+3}) \\ 0 \end{array}$$

The height is $x^2 - x + 1.$

53. a. $R(40) = 3000 \Rightarrow R(40) - 3000 = 0$ and $R(60) = 3000 \Rightarrow R(60) - 3000 = 0.$ Thus, 40 and 60 are zeros of $R(x) - 3000.$ Therefore,

$$\begin{aligned} R(x) - 3000 &= a(x-40)(x-60) \\ &= a\left(x^2 - 100x + 2400\right) \end{aligned}$$

Since (30, 2400) lies on $R(x)$, we have
$2400 - 3000 = a(30-40)(30-60) \Rightarrow$
$-600 = a(-10)(-30) \Rightarrow a = -2.$
Thus

$$\begin{aligned} R(x) - 3000 &= -2(x-40)(x-60) \\ &= -2x^2 + 200x - 4800 \end{aligned}$$

b. $R(x) - 3000 = -2x^2 + 200x - 4800 \Rightarrow$
$R(x) = -2x^2 + 200x - 1800$

c. The maximum weekly revenue occurs at the vertex of the function, $\left(-\dfrac{b}{2a}, R\left(-\dfrac{b}{2a}\right)\right).$

$$-\frac{b}{2a} = -\frac{200}{2(-2)} = 50$$

$$R(50) = -2(50)^2 + 200(50) - 1800 = 3200$$

The maximum revenue is \$3200 if the phone is priced at \$50.

54. a. $R(8) = 4725 \Rightarrow R(8) - 4725 = 0.$ and $R(12) = 4725 \Rightarrow R(12) - 4725 = 0.$ Thus, 8 and 12 are zeros of $R(x) - 4725.$

Therefore, $R(x) - 4725 = a(x-8)(x-12) = a\left(x^2 - 20x + 96\right)$

Since (6, 4125) lies on $R(x)$, we have $4125 - 4725 = a(6-8)(6-12) \Rightarrow -600 = a(-2)(-6) \Rightarrow a = -50$

Thus, $R(x) - 4725 = -50(x-8)(x-12) = -50x^2 + 1000x - 4800$

b. $R(x) - 4725 = -50x^2 + 1000x - 4800 \Rightarrow R(x) = -50x^2 + 1000x - 75$

c. The maximum weekly revenue occurs at the vertex of the function, $\left(-\frac{b}{2a}, R\left(-\frac{b}{2a}\right)\right).$

$$-\frac{b}{2a} = -\frac{1000}{2(-50)} = 10$$

$R(0) = -50(10)^2 + 1000(10) - 75 = 4925$

The maximum revenue is \$4925 if tickets are priced at \$10 each.

55. Since $t = 11$ represents 2002, we have $C(11) = 97.6 \Rightarrow C(x) = (x-11)Q(x) + 97.6 \Rightarrow$ $C(t) - 97.6 = (t-11)Q(t).$ So

$$\begin{aligned} -0.0006t^3 - 0.0613t^2 + 2.0829t + 82.904 - 97.6 &= -0.0006t^3 - 0.0613t^2 + 2.0829t - 14.6960 \\ &= (x-11)Q(x). \end{aligned}$$

Use synthetic division to find $Q(x)$:

$$\begin{array}{r|rrrr} 11 & -0.0006 & -0.0613 & 2.0829 & -14.6960 \\ & & -0.0066 & -0.7469 & 14.6960 \\ \hline & -0.0006 & -0.0679 & 1.3360 & 0 \end{array}$$

Now solve the depressed equation to find another zero:

$$-0.0006^2 - 0.0679t + 1.3360 = 0 \Rightarrow t = \frac{0.0679 \pm \sqrt{0.0679^2 - 4(-0.0006)(1.3360)}}{2(-0.0006)} \Rightarrow$$

$$t = \frac{0.0679 \pm \sqrt{0.00781681}}{-0.0012} = \frac{0.0679 \pm 0.0884}{-0.0012} \Rightarrow t \approx -130.26 \text{ or } t \approx 17.0939.$$

Since we must find t greater than 0, $t = 17$, and the year is 1991 + 17 = 2008.

56. Since $t = 1$ represents 1999, we have $s(1) = 737.7 \Rightarrow s(t) = (t-1)Q(t) + 737.7 \Rightarrow$ $s(t) - 737.7 = (t-1)Q(t).$ So

$$\begin{aligned} -0.1779t^3 - 2.8292t^2 + 42.0240t + 698.6831 - 737.7 &= -0.1779t^3 - 2.8292t^2 + 42.0240t - 39.0169 \\ &= (t-1)Q(t) \end{aligned}$$

Use synthetic division to find $Q(t)$:

$$\begin{array}{r|rrrr} 1 & -0.1779 & -2.8292 & 42.0240 & -39.0169 \\ & & -0.1779 & -3.0071 & 39.0169 \\ \hline & -0.1779 & -3.0071 & 39.0169 & 0 \end{array}$$

Now solve the depressed equation to find another zero:

$$-0.1779t^2 - 3.0071t + 39.0169 = 0 \Rightarrow t = \frac{3.0071 \pm \sqrt{(-3.0071)^2 - 4(-0.1779)(39.0169)}}{2(-0.1779)} \Rightarrow$$

$$t = \frac{3.0071 \pm \sqrt{36.8071}}{-0.3558} \Rightarrow t \approx -25.5030 \text{ or } t \approx 8.5997$$

Since t must be positive, $t \approx 9$, and the year is 1998 + 9 = 2007.

57. $M(t) = -0.0027t^3 + 0.3681t^2 - 5.8645t + 195.2782$

Since $M(2) = 191.736,\ M(t) = (t-2)Q(t) + 195.2782 \Rightarrow M(t) - 195.2782 = (t-2)Q(t)$

We must find two other zeros of

$$F(t) = M(t) - 185 = -0.0027t^3 + 0.3681t^2 - 5.8645t + 195.2782 - 185$$
$$= -0.0027t^3 + 0.3681t^2 - 5.8645t + 10.2782$$

Because 2 is a zero of *F*(*t*), use synthetic division to find *Q*(*t*).

$$\begin{array}{r|rrrr} \underline{2} & -0.0027 & 0.3681 & -5.8645 & 10.2782 \\ & & -0.0054 & 0.7254 & -10.2782 \\ \hline & -0.0027 & 0.3627 & -5.1391 & 0 \end{array}$$

Now use the quadratic formula to solve the depressed equation.

$$t = \frac{-0.3627 \pm \sqrt{0.3627^2 - 4(-0.0027)(-5.1391)}}{2(-0.0027)} = \frac{-0.3627 \pm \sqrt{0.0760}}{-0.0054} \Rightarrow t \approx 16.1 \text{ or } t \approx 118$$

Thus, the model shows that the Marine Corps had about 186,000 when $t \approx 16$, or in the year 1990 + 16 = 2006.

58. $U(t) = -0.0374t^3 + 0.5934t^2 - 2.0553t + 6.7478$

Since $U(6) = 7.7,\ U(t) = (t-6)Q(x) + 7.7 \Rightarrow U(t) - 7.7 = (t-6)Q(6)$

We must find two other zeros of

$$F(t) = U(t) - 7.7 = -0.0374t^3 + 0.5934t^2 - 2.0553t + 6.7478 - 7.7$$
$$= -0.0374t^3 + 0.5934t^2 - 2.0553t - 0.9522$$

Because 6 is a zero of *F*(*t*), use synthetic division to find *Q*(*t*).

$$\begin{array}{r|rrrr} \underline{6} & -0.0374 & 0.5934 & -2.0553 & -0.9522 \\ & & -0.2244 & 2.2140 & 0.9522 \\ \hline & -0.0374 & 0.3690 & 0.1587 & 0 \end{array}$$

Now use the quadratic formula to solve the depressed equation.

$$t = \frac{-0.3690 \pm \sqrt{0.3690^2 - 4(-0.0374)(0.1587)}}{2(-0.0374)} = \frac{-0.3690 \pm \sqrt{0.1599}}{-0.0748} \Rightarrow t \approx -0.41 \text{ or } t \approx 10.3$$

Reject the negative solution.

According to the model, there was a 7.7% unemployment level in 2002 + 10 = 2012.

3.3 Beyond the Basics

59. Divide $4x^3 + 8x^2 - 11x + 3$ by $\left(x - \frac{1}{2}\right)$:

$$\begin{array}{r|rrrr} \frac{1}{2} & 4 & 8 & -11 & 3 \\ & & 2 & 5 & -3 \\ \hline & 4 & 10 & -6 & 0 \end{array}$$

Now divide $4x^2 + 10x - 6$ by $\left(x - \frac{1}{2}\right)$:

$$\begin{array}{r|rrr} \frac{1}{2} & 4 & 10 & -6 \\ & & 2 & 6 \\ \hline & 4 & 12 & 0 \end{array}$$

Since $\left(x - \frac{1}{2}\right)$ does not divide $4x + 12$, $\left(x - \frac{1}{2}\right)$ is a root of multiplicity 2 of $4x^3 + 8x^2 - 11x + 3$.

60. Divide $9x^3 + 3x^2 - 8x - 4$ by $\left(x + \frac{2}{3}\right)$:

$$\begin{array}{r|rrrr} -\frac{2}{3} & 9 & 3 & -8 & -4 \\ & & -6 & 2 & 4 \\ \hline & 9 & -3 & -6 & 0 \end{array}$$

Now divide $9x^2 - 3x - 6$ by $\left(x + \frac{2}{3}\right)$:

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(*continued*)

$$\begin{array}{r|rrr} -\frac{2}{3} & 9 & -3 & -6 \\ & & -6 & 6 \\ \hline & 9 & -9 & 0 \end{array}$$

Since $\left(x+\frac{2}{3}\right)$ does not divide $9x-9$, $\left(x+\frac{2}{3}\right)$ is a root of multiplicity 2 of $9x^3+3x^2-8x-4$.

61. a. $x+a$ is a factor of x^n+a^n if *n* is an odd integer. The possible rational zeros of x^n+a^n are $\{\pm a, \pm a^2, \pm a^3, \ldots, \pm a^n\}$. Since $x+a$ is a factor means that $-a$ is a root, then $(-a)^n+a^n=0 \Rightarrow (-a)^n=-a^n$ only for odd values of *n*.

b. $x+a$ is a factor of x^n-a^n if *n* is an even integer. The possible rational zeros of x^n-a^n are $\{\pm a, \pm a^2, \pm a^3, \ldots, \pm a^n\}$. Since $x+a$ is a factor means that $-a$ is a root, then $(-a)^n-a^n=0 \Rightarrow (-a)^n=a^n$ only for even values of *n*.

c. There is no value of *n* for which $x-a$ is a factor of x^n+a^n. The possible rational zeros of x^n+a^n are $\{\pm a, \pm a^2, \pm a^3, \ldots, \pm a^n\}$. If $x-a$ is a factor, then *a* is a root, and $a^n+a^n=0 \Rightarrow a^n=-a^n$, which is not possible.

d. $x-a$ is a factor of x^n-a^n for all positive integers *n*. The possible rational zeros of x^n-a^n are $\{\pm a, \pm a^2, \pm a^3, \ldots, \pm a^n\}$. If $x-a$ is a factor, then *a* is a root, and $a^n-a^n=0$, which is true for all values of *n*. However, if *n* is negative, then $x^n-a^n=\frac{1}{x^{-n}}-\frac{1}{a^{-n}}$ and $x-a$ is not a factor.

62. a. According to exercise 61d, $x-a$ is a factor of x^n-a^n for any positive integer *n*. Using this result with *x* = 7, *a* = 2, and *n* = 11, we find that 7 – 2 = 5 is a factor of $7^{11}-2^{11}$. Therefore, $7^{11}-2^{11}$ is divisible by 5.

b. $19^{20}-10^{20}=\left(19^{10}-10^{10}\right)\left(19^{10}+10^{10}\right)$. According to exercises 61b and 61d, $x+a$ is a factor of x^n-a^n if *n* is an odd integer, and $x-a$ is a factor of x^n-a^n for any positive integer *n*. Using these results with *x* = 19, *a* = 10, and *n* = 10, we find that 19 – 10 = 9 and 19 + 10 = 29 are both factors of $19^{10}-10^{10}$,and therefore, $19^{20}-10^{20}$ is divisible by 9 and 29. Since 9 and 29 have no common factors, $19^{20}-10^{20}$ must be divisible by their product, 9(29) = 261.

63. a. Divide the divisor and the dividend by 2 so that the leading coefficient of the divisor is 1:

$$\frac{2x^3+3x^2+6x-2}{2x-1}=\frac{2x^3+3x^2+6x-2}{2\left(x-\frac{1}{2}\right)}=\frac{x^3+\frac{3}{2}x^2+3x-1}{\left(x-\frac{1}{2}\right)}$$

$$\begin{array}{r|rrrr} \frac{1}{2} & 1 & \frac{3}{2} & 3 & -1 \\ & & \frac{1}{2} & 1 & 2 \\ \hline & 1 & 2 & 4 & 1 \end{array}$$

Because the original polynomials were divided by 2 to use synthetic division, multiply the remainder by 2 to find the remainder for the original division, 2.

b. Divide the divisor and the dividend by 2 so that the leading coefficient of the divisor is 1:

$$\frac{2x^3-x^2-4x+1}{2x+3}=\frac{2x^3-x^2-4x+1}{2\left(x+\frac{3}{2}\right)}=\frac{x^3-\frac{1}{2}x^2-2x+\frac{1}{2}}{\left(x+\frac{3}{2}\right)}$$

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(*continued*)

$$\begin{array}{r|rrrr} -\frac{3}{2} & 1 & -\frac{1}{2} & -2 & \frac{1}{2} \\ & & -\frac{3}{2} & 3 & -\frac{3}{2} \\ \hline & 1 & -2 & 1 & -1 \end{array}$$

Because the original polynomials were divided by 2 to use synthetic division, multiply the remainder by 2 to find the remainder for the original division, –2.

64. a.

$$\begin{array}{r|rrrr} -c & 3 & -5c & -3c^2 & 3c^3 \\ & & -3c & 8c^2 & -5c^3 \\ \hline & 3 & -8c & 5c^2 & -2c^3 \end{array}$$

The remainder is $-2c^3$ so $g(-c) = -2c^3$.

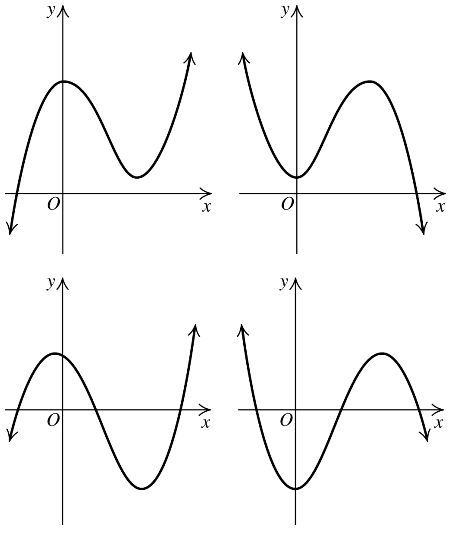
b.

$$\begin{array}{r|rrrr} 2c & 3 & -5c & -3c^2 & 3c^3 \\ & & 6c & 2c^2 & -2c^3 \\ \hline & 3 & c & -c^2 & c^3 \end{array}$$

The remainder is c^3 so $g(2c) = c^3$.

65. $f(x) = ax^3 + bx^2 + cx + d,\ a \neq 0$

There are two possibilities for the end behavior of *f*:



$y \to -\infty$ as $x \to -\infty$
$y \to \infty$ as $x \to \infty$

$y \to \infty$ as $x \to -\infty$
$y \to -\infty$ as $x \to \infty$

Therefore, there is one sign change from the left end of *f* to the right side of *f* or there are three sign changes.

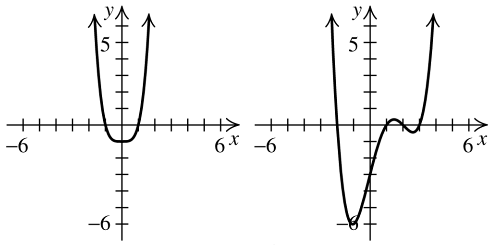
By the Intermediate Value Theorem, then, there is one zero (if there is one sign change) or there are three zeros (if there are three sign changes.)

66. $g(x) = kx^4 + lx^3 + mx^2 + nx + p,\ k \neq 0$

$kp < 0$ means that *k* and *p* have opposite signs. If $p < 0$, the *y*-intercept is below the *x*-axis, and if $k > 0$, the end behavior is $g(x) \to \infty$ as $x \to -\infty$ and $g(x) \to \infty$ as $x \to \infty$. Therefore, there are at least 2 real zeros, r_1 and r_2. Thus,

$g(x) = (x - r_1)(x - r_2)(x^2 + sx + t)$.

The quadratic factor has either 0 or 2 real zeros. Therefore *g* has either 2 or 4 real zeros. Two examples are shown below.

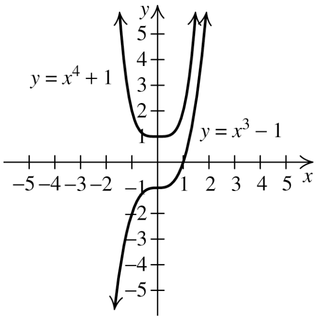


$y = x^4 - 1$ $\qquad$ $y = \frac{1}{4}(x^4 - 4x^3 - x^2 + 16x - 12)$

Similar reasoning applies if $p > 0$ and $k < 0$.

67. $dp < 0$ means that *d* and *p* have opposite signs, and, thus, are on opposite sides of the *x*-axis. If $d > 0$ then $p < 0$. Since $kp < 0$, $k > 0$ also. The end behavior is $g(x) \to \infty$ as $x \to -\infty$ and $g(x) \to \infty$ as $x \to \infty$. However, *f* has end behavior shown in exercise 65. Therefore, the two graphs must intersect. Similar reasoning applies if $d < 0$.
Algebraic proof: Assume $d > 0$ and $p < 0$. Consider the fourth degree polynomial $g - f$. Then $(g - f)(0) = p - d < 0$. Since $k > 0$, $(g - f)(x)$ has upward end behavior on both ends, so $(g - f)(x)$ is eventually > 0. By the Intermediate Value Theorem, there is some *c* such that $(g - f)(c) = 0 \Rightarrow g(c) = f(c)$. Therefore, the graphs intersect.

68. Answers will vary. One example is $f(x) = x^3 - 1$ and $g(x) = x^4 + 1$.



$d < 0, k > 0, p > 0$

If the graphs intersect, then

$f(x) = g(x) \Rightarrow x^3 - 1 = x^4 + 1 \Rightarrow$

$x^3 - x^4 = 2 \Rightarrow x^3(1 - x) = 2$

From $x^3(1-x) > 0,$ we have (1)

$x^3 < 0 \Rightarrow x < 0$ and $1 - x < 0 \Rightarrow 1 < x$ or (2)

$x^3 > 0 \Rightarrow x > 0$ and $1 - x > 0 \Rightarrow x < 1.$

Examining choice (1), we have a contradiction, *x* cannot be less than 0 and greater than 1 at the same time. Examining choice (2), we have $0 < x < 1 \Rightarrow 0 < x^4 < 1$ and $0 < x^3 < 1 \Rightarrow 0 < x^4 < 1$ and $-1 < -x^3 < 0 \Rightarrow -1 < x^4 - x^3 < 1$ which contradicts $x^3 - x^4 = 2.$

69. Answers will vary. One example is $f(x) = x^4 + 2$

3.3 Critical Thinking/Discussion/Writing

70. a. The quotient when *F*(*x*) is divided by 2*x* – 6 is one-half the quotient when *F*(*x*) is divided by *x* – 3, but the remainders are identical.

b. If $\dfrac{F(x)}{x+\frac{b}{a}} = Q(x) + \dfrac{R(x)}{x+\frac{b}{a}},$ then

$$\frac{F(x)}{ax+b} = \frac{1}{a}\left[\frac{F(x)}{x+\frac{b}{a}}\right] = \frac{1}{a}\left[Q(x) + \frac{R(x)}{x+\frac{b}{a}}\right] = \frac{1}{a}Q(x) + \frac{R(x)}{ax+b}$$

So, the remainders are again identical and the quotient when *F*(*x*) is divided by *ax* + *b* is $\frac{1}{a}$ times the quotient when *F*(*x*) is divided by $x + \frac{b}{a}.$

3.3 Maintaining Skills

71.

1⌋	1	−1	2	1	−3
		1	0	2	3
	1	0	2	3	0

The quotient is $x^3 + 2x + 3$ and the remainder is 0.

72.

−2⌋	1	1	−1	2
		−2	2	−2
	1	−1	1	0

The quotient is $x^2 - x + 1$ and the remainder is 0.

73.

5⌋	−1	5	0	0	2	−10	3
		−5	0	0	0	10	0
	−1	0	0	0	2	0	3

The quotient is $-x^5 + 2x$ and the remainder is 3.

74.

−1⌋	1	3	3	2	7
		−1	−2	−1	−1
	1	2	1	1	6

The quotient is $x^3 + 2x^2 + x + 1$ and the remainder is 6.

75. The factors of 11 are $\pm 1, \pm 11.$

76. The factors of 18 are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9,$ and $\pm 18.$

77. The factors of 51 are $\pm 1, \pm 3, \pm 17, \pm 51.$

78. The factors of 72 are
$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8,$
$\pm 9, \pm 12, \pm 18, \pm 24, \pm 36, \pm 72$

79. $x^2 + 2x - 1$
Degree: 2
Leading coefficient: 1
Constant term: –1

80. $-x^3 + x + 7$
Degree: 3
Leading coefficient: –1
Constant term: 7

81. $-7x^{10}+3x^3-2x+4$
Degree: 10
Leading coefficient: −7
Constant term: 4

82. $13x^9+2x^{15}+3x-11$
Degree: 15
Leading coefficient: 2
Constant term: −11

3.4 The Real Zeros of a Polynomial Function

3.4 Practice Problems

1. $F(x)=2x^3+3x^2-6x-8$
The factors of the constant term, −8, are $\{\pm1,\pm2,\pm4,\pm8\}$, and the factors of the leading coefficient, 2, are $\{\pm1,\pm2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2},\pm1,\pm2,\pm4,\pm8\right\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} -2 & 2 & 3 & -6 & -8 \\ & & -4 & 2 & 8 \\ \hline & 2 & -1 & -4 & 0 \end{array}$$

The remainder is 0, so −2 is a zero of the function.

$$2x^3+3x^2-6x-8=(x+2)\left(2x^2-x-4\right)$$

Now find the zeros of $2x^2-x-4$ using the quadratic formula:

$$x=\frac{1\pm\sqrt{(-1)^2-4(2)(-4)}}{2(1)}=\frac{1\pm\sqrt{33}}{2}$$, which are not rational roots.
The only rational zero is {−2}.

2. $2x^3-9x^2+6x-1=0$
The factors of the constant term, −1, are $\{\pm1\}$, and the factors of the leading coefficient, 2, are $\{\pm1,\pm2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2},\pm1\right\}$. Use synthetic division to find one rational zero:

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -9 & 6 & -1 \\ & & 1 & -4 & 1 \\ \hline & 2 & -8 & 2 & 0 \end{array}$$

Thus, $2x^3-9x^2+6x-1=0\Rightarrow$
$\left(x-\frac{1}{2}\right)\left(2x^2-8x+2\right)=0\Rightarrow$
$x-\frac{1}{2}=0\Rightarrow x=\frac{1}{2}$ or
$2x^2-8x+2=0$

$$x=\frac{-(-8)\pm\sqrt{(-8)^2-4(2)(2)}}{2(2)}=\frac{8\pm\sqrt{48}}{4}$$
$$=2\pm\sqrt{3}$$

Solution set: $\left\{\frac{1}{2},\ 2-\sqrt{3},\ 2+\sqrt{3}\right\}$

3. $f(x)=2x^5+3x^2+5x-1$
There is one sign change in $f(x)$, so there is one positive zero.

$$\begin{aligned} f(-x)&=2(-x)^5+3(-x)^2+5(-x)-1\\ &=-2x^5+3x^2-5x-1\end{aligned}$$

There are two sign changes in $f(-x)$, so there are either 2 or 0 negative zeros.

4. $f(x)=2x^3+5x^2+x-2$
The possible rational zeros are $\left\{\pm\frac{1}{2},\pm1,\pm2\right\}$.
There is one sign change in $f(x)$, so there is one positive zero.

$$\begin{aligned} f(-x)&=2(-x)^3+5(-x)^2+(-x)-2\\ &=-2x^3+5x^2-x-2\end{aligned}$$

There are two sign changes in $f(-x)$, so there are either 2 or 0 negative zeros. Try synthetic division by $x-k$ with $k=1, 2, 3, \ldots$. The first integer that makes each number in the last row a 0 or a positive number is an upper bound on the zeros of $F(x)$. Then use synthetic division by $x-k$ with $k=-1, -2, -3, \ldots$. The first negative integer for which the numbers in the last row alternate in sign is a lower bound on the zeros of $F(x)$. In this case, 1 is an upper bound and −3 is a lower bound.

$$\begin{array}{r|rrrr} 1 & 2 & 5 & 1 & -2 \\ & & 2 & 7 & 8 \\ \hline & 2 & 7 & 8 & 6 \end{array} \qquad \begin{array}{r|rrrr} -3 & 2 & 5 & 1 & -2 \\ & & -6 & 3 & -12 \\ \hline & 2 & -1 & 4 & -14 \end{array}$$

5. $f(x)=3x^4-11x^3+22x-12$
Step 1: *f* has at most 4 real zeros.
Step 2: $f(x)=3x^4-11x^3+0x^2+22x-12$
There are three sign changes in *f*, so *f* has 1 or 3 positive zeros.

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$$\begin{aligned} f(-x) &= 3(-x)^4 - 11(-x)^3 + 0(-x)^2 \\ &\qquad + 22(-x) - 12 \\ &= 3x^4 + 11x^3 + 0x^2 - 22x - 12 \end{aligned}$$

There is one sign change in $f(-x)$, so f has 1 negative zero.

Step 3: The factors of the constant term, −12, are $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$, and the factors of the leading coefficient, 3, are $\{\pm 1, \pm 3\}$. The possible rational zeros are $\left\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}\right\}$.

Steps 4−7: Test for zeros until a zero or an upper bound is found. Try a positive possibility:

$$\begin{array}{r|rrrrr} 3 & 3 & -11 & 0 & 22 & -12 \\ & & 9 & -6 & -18 & 12 \\ \hline & 3 & -2 & -6 & 4 & 0 \end{array}$$

Since 3 is a zero, we have $f(x) = (x-3)(3x^3 - 2x^2 - 6x + 4)$.

Now find the a zero of $Q_1(x) = 3x^3 - 2x^2 - 6x + 4$. The factors of the constant term, 4, are $\{\pm 1, \pm 2, \pm 4\}$, and the factors of the leading coefficient, 3, are $\{\pm 1, \pm 3\}$. The possible rational zeros are $\left\{\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}\right\}$. Try the positive possibilities first.

$$\begin{array}{r|rrrr} \frac{2}{3} & 3 & -2 & -6 & 4 \\ & & 2 & 0 & -4 \\ \hline & 3 & 0 & -6 & 0 \end{array}$$

Thus, $f(x) = (x-3)\left(x - \frac{2}{3}\right)(3x^2 - 6)$.

Now solve $3x^2 - 6 = 0$.

$3x^2 - 6 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Solution set: $\left\{\frac{2}{3}, 3, \sqrt{2}, -\sqrt{2}\right\}$

6. $f(x) = 3x^3 - x^2 - 9x + 3$

Step 1: Since the degree, 3, is odd, and the leading coefficient, 3, is positive, the end behavior is similar to that of $y = x^3$.

Step 2: Solve $3x^3 - x^2 - 9x + 3 = 0$ to find the real zeros. There are two sign changes in f, so f has either 2 or 0 positive zeros.

$$\begin{aligned} f(-x) &= 3(-x)^3 - (-x)^2 - 9(-x) + 3 \\ &= -3x^3 + x^2 + x + 3 \end{aligned}$$

There is one sign change in $f(-x)$, so there is one negative zero.

By the Rational Root Theorem, the possible rational roots are $\left\{\pm 1, \pm 3, \pm \frac{1}{3}\right\}$. Trying each value, we find that $\frac{1}{3}$ is a rational zero.

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & -9 & 3 \\ & & 1 & 0 & -3 \\ \hline & 3 & 0 & -9 & 0 \end{array}$$

Solve the depressed equation $3x^2 - 9 = 0$ to find the remaining zeros.

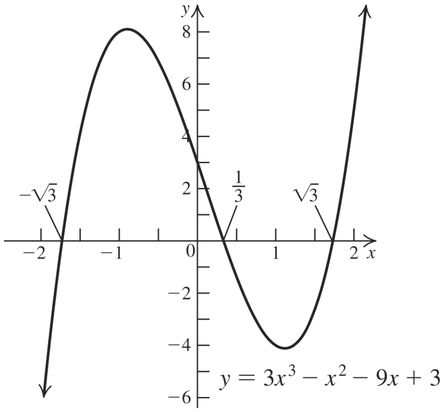
$3x^2 - 9 = 0 \Rightarrow 3x^2 = 9 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

Step 3: The three zeros divide the x-axis into four intervals, $(-\infty, -\sqrt{3})$, $\left(-\sqrt{3}, \frac{1}{3}\right)$, $\left(\frac{1}{3}, \sqrt{3}\right)$, and $(\sqrt{3}, \infty)$.

Step 4: Determine the sign of a test value in each interval

Interval	Test point	Value of $f(x)$	Above/below x-axis
$(-\infty, -\sqrt{3})$	−2	−7	below
$\left(-\sqrt{3}, \frac{1}{3}\right)$	0	3	above
$\left(\frac{1}{3}, \sqrt{3}\right)$	1	−4	below
$(\sqrt{3}, \infty)$	2	5	above

Plot the zeros, y-intercepts, and test points, and then join the points with a smooth curve.



3.4 Basic Concepts and Skills

1. Let $P(x) = a_n x^n + \ldots + a_0$ with integer coefficients. If $\frac{p}{q}$ is a rational zero of $P(x)$, then $\frac{p}{q} = \frac{\text{possible factors of } \underline{a_0}}{\text{possible factors of } \underline{a_n}}$.

2. If the terms of a polynomial are written in descending order, then a variation of sign occurs when the signs of two <u>consecutive</u> terms differ.

3. The number of positive zeros of a polynomial function $P(x)$, is equal to the number of <u>variations</u> of sign of $P(x)$ or less than that number by <u>an even integer</u>.

4. If a polynomial $P(x)$ has no zero greater than a number k, then k is called <u>an upper bound</u> of the zeros of $P(x)$. If $P(x)$ has no zero less than a number m, then m is called <u>a lower bound</u> of the zeros of $P(x)$.

5. False. Possible rational zeros of $P(x) = 9x^3 - 9x^2 - x + 1$ are $\left\{\pm\frac{1}{9}, \pm\frac{1}{3}, \pm 1\right\}$.

6. True. There is one variation of sign, so there is exactly one real zero.

7. The factors of the constant term, 5, are $\{\pm 1, \pm 5\}$, and the factors of the leading coefficient, 3, are $\{\pm 1, \pm 3\}$. The possible rational zeros are $\left\{\pm\frac{1}{3}, \pm 1, \pm\frac{5}{3}, \pm 5\right\}$.

8. The factors of the constant term, 1, are $\{\pm 1\}$, and the factors of the leading coefficient, 2, are $\{\pm 1, \pm 2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm 1\right\}$.

9. The factors of the constant term, 6, are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$, and the factors of the leading coefficient, 4, are $\{\pm 1, \pm 2, \pm 4\}$.The possible rational zeros are $\left\{\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3, \pm 6\right\}$.

10. The factors of the constant term, –35, are $\{\pm 1, \pm 5, \pm 7, \pm 35\}$, and the factors of the leading coefficient, 6, are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. The possible rational zeros are

$$\left\{\pm\frac{1}{6}, \pm\frac{1}{2}, \pm 1, \pm\frac{5}{3}, \pm\frac{5}{2}, \pm 5, \pm 7, \pm\frac{35}{2}, \pm\frac{1}{3}, \pm\frac{5}{6}, \pm\frac{7}{6}, \pm\frac{7}{3}, \pm\frac{7}{2}, \pm\frac{35}{6}, \pm\frac{35}{3}, \pm 35\right\}.$$

11. The factors of the constant term, 4, are $\{\pm 1, \pm 2, \pm 4\}$, and the factors of the leading coefficient, 1, are $\{\pm 1\}$. The possible rational zeros are $\{\pm 1, \pm 2, \pm 4\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} \underline{1} & 1 & -1 & -4 & 4 \\ & & 1 & 0 & -4 \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

So,

$$\begin{aligned} x^3 - x^2 - 4x + 4 &= (x-1)(x^2-4) \\ &= (x-1)(x-2)(x+2) \Rightarrow \end{aligned}$$

the rational zeros are {–2, 1, 2}.

12. The factors of the constant term, 6, are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$, and the factors of the leading coefficient, 1, are $\{\pm 1\}$. The possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} \underline{-1} & 1 & -4 & 1 & 6 \\ & & -1 & 5 & -6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$\begin{aligned} x^3 - 4x^2 + x + 6 &= (x+1)(x^2-5x+6) \\ &= (x+1)(x-2)(x-3) \Rightarrow \end{aligned}$$

the rational zeros are {–1, 2, 3}.

13. The factors of the constant term, 2, are $\{\pm 1, \pm 2\}$, and the factors of the leading coefficient, 1, are $\{\pm 1\}$. The possible rational zeros are $\{\pm 1, \pm 2\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} \underline{-1} & 1 & 1 & 2 & 2 \\ & & -1 & 0 & -2 \\ \hline & 1 & 0 & 2 & 0 \end{array}$$

So, $x^3 + x^2 + 2x + 2 = (x+1)(x^2+2) \Rightarrow$ the rational zero is {–1}.

14. The factors of the constant term, 6, are $\{\pm1, \pm2, \pm3, \pm6\}$, and the factors of the leading coefficient, 1, are $\{\pm1\}$. The possible rational zeros are $\{\pm1, \pm2, \pm3, \pm6\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 2 & 6 \\ & & -3 & 0 & -6 \\ \hline & 1 & 0 & 2 & 0 \end{array}$$

So, $x^3 + 3x^2 + 2x + 6 = (x + 3)(x^2 + 2) \Rightarrow$ the rational zero is $\{-3\}$.

15. The factors of the constant term, 6, are $\{\pm1, \pm2, \pm3, \pm6\}$, and the factors of the leading coefficient, 2, are $\{\pm1, \pm2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm2, \pm3, \pm6\right\}$.
Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$$\begin{aligned} 2x^3 + x^2 - 13x + 6 &= (x - 2)(2x^2 + 5x - 3) \\ &= (x - 2)(x + 3)(2x - 1) \Rightarrow \end{aligned}$$

the rational zeros are $\left\{-3, \frac{1}{2}, 2\right\}$.

16. The factors of the constant term, –2, are $\{\pm1, \pm2\}$, and the factors of the leading coefficient, 6, are $\{\pm1, \pm2, \pm3, \pm6\}$. The possible rational zeros are $\left\{\pm\frac{1}{6}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm1, \pm2\right\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} -2 & 6 & 13 & 1 & -2 \\ & & -12 & -2 & 2 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

$$\begin{aligned} 6x^3 + 13x^2 + x - 2 &= (x + 2)(6x^2 + x - 1) \\ &= (x + 2)(2x + 1)(3x - 1) \Rightarrow \end{aligned}$$

the rational zeros are $\left\{-2, -\frac{1}{2}, \frac{1}{3}\right\}$.

17. The factors of the constant term, –2, are $\{\pm1, \pm2\}$, and the factors of the leading coefficient, 3, are $\{\pm1, \pm3\}$. The possible rational zeros are $\left\{\pm\frac{1}{3}, \pm\frac{2}{3}, \pm1, \pm2\right\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} \frac{2}{3} & 3 & -2 & 3 & -2 \\ & & 2 & 0 & 2 \\ \hline & 3 & 0 & 3 & 0 \end{array}$$

$3x^3 - 2x^2 + 3x - 2 = \left(x - \frac{2}{3}\right)(3x^2 + 3) \Rightarrow$ the rational zero is $\left\{\frac{2}{3}\right\}$.

18. The factors of the constant term, 6, are $\{\pm1, \pm2, \pm3, \pm6\}$, and the factors of the leading coefficient, 2, are $\{\pm1, \pm2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm2, \pm3, \pm6\right\}$.
Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} -\frac{3}{2} & 2 & 3 & 4 & 6 \\ & & -3 & 0 & -6 \\ \hline & 2 & 0 & 4 & 0 \end{array}$$

$2x^3 + 3x^2 + 4x + 6 = \left(x + \frac{3}{2}\right)(2x^2 + 4) \Rightarrow$ the rational zero is $\left\{-\frac{3}{2}\right\}$.

19. The factors of the constant term, 2, are $\{\pm1, \pm2\}$, and the factors of the leading coefficient, 3, are $\{\pm1, \pm3\}$. The possible rational zeros are $\left\{\pm\frac{1}{3}, \pm\frac{2}{3}, \pm1, \pm2\right\}$.

Use synthetic division to find one rational root:

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & 7 & 8 & 2 \\ & & -1 & -2 & -2 \\ \hline & 3 & 6 & 6 & 0 \end{array}$$

$3x^3 + 7x^2 + 8x + 2 = \left(x + \frac{1}{3}\right)(3x^2 + 6x + 6) \Rightarrow$ the rational zero is $\left\{-\frac{1}{3}\right\}$.

20. The factors of the constant term, 4, are $\{\pm1, \pm2, \pm4\}$, and the factors of the leading coefficient, 2, are $\{\pm1, \pm2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm1, \pm2, \pm4\right\}$. Use synthetic division to find one rational root:

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(*continued*)

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 1 & 8 & 4 \\ & & -1 & 0 & -4 \\ \hline & 2 & 0 & 8 & 0 \end{array}$$

$2x^3 + x^2 + 8x + 4 = \left(x + \frac{1}{2}\right)(2x^2 + 8) \Rightarrow$ the rational zero is $\left\{-\frac{1}{2}\right\}$.

21. The factors of the constant term, –2, are $\{\pm 1, \pm 2\}$, and the factors of the leading coefficient, 1, are $\{\pm 1\}$. The possible rational zeros are $\{\pm 1, \pm 2\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & -1 & -1 & -2 \\ & & -1 & 2 & -1 & 2 \\ \hline & 1 & -2 & 1 & -2 & 0 \end{array}$$

So, –1 is a rational zero. Use synthetic division again to find another rational zero:

$$x^4 - x^3 - x^2 - x - 2 = (x+1)(x^3 - 2x^2 + x - 2)$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 1 & -2 \\ & & 2 & 0 & 2 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

$x^4 - x^3 - x^2 - x - 2 = (x+1)(x-2)(x^2+1) \Rightarrow$ the rational zeros are {–1, 2}.

22. The factors of the constant term, –4, are $\{\pm 1, \pm 2, \pm 4\}$, and the factors of the leading coefficient, 2, are $\{\pm 1, \pm 2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm 1, \pm 2, \pm 4\right\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrrr} -1 & 2 & 3 & 8 & 3 & -4 \\ & & -2 & -1 & -7 & 4 \\ \hline & 2 & 1 & 7 & -4 & 0 \end{array}$$

So, –1 is a rational zero. Use synthetic division again to find another rational zero:

$$\begin{aligned} &2x^4 + 3x^3 + 8x^2 + 3x - 4 \\ &\qquad = (x+1)(2x^3 + x^2 + 7x - 4) \end{aligned}$$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & 1 & 7 & -4 \\ & & 1 & 1 & 4 \\ \hline & 2 & 2 & 8 & 0 \end{array}$$

$$\begin{aligned} &2x^4 + 3x^3 + 8x^2 + 3x - 4 \\ &\qquad = (x+1)\left(x - \frac{1}{2}\right)(2x^2 + 2x + 8) \\ &\qquad = 2(x+1)\left(x - \frac{1}{2}\right)(x^2 + x + 4) \Rightarrow \end{aligned}$$

the rational zeros are $\left\{-1, \frac{1}{2}\right\}$.

23. The factors of the constant term, 12, are $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$, and the factors of the leading coefficient, 1, are $\{\pm 1\}$. The possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrrr} -3 & 1 & -1 & -13 & 1 & 12 \\ & & -3 & 12 & 3 & -12 \\ \hline & 1 & -4 & -1 & 4 & 0 \end{array}$$

So, –3 is a rational zero. Use synthetic division again to find another rational zero:

$$\begin{aligned} &x^4 - x^3 - 13x^2 + x + 12 \\ &\qquad = (x+3)(x^3 - 4x^2 - x + 4) \end{aligned}$$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & -1 & 4 \\ & & -1 & 5 & -4 \\ \hline & 1 & -5 & 4 & 0 \end{array}$$

So, –1 is also a rational zero.

$$\begin{aligned} &x^4 - x^3 - 13x^2 + x + 12 \\ &\qquad = (x+3)(x+1)(x^2 - 5x + 4) \\ &\qquad = (x+3)(x+1)(x-4)(x-1) \Rightarrow \end{aligned}$$

the rational zeros are {–3, –1, 1, 4}.

24. The factors of the constant term, –2, are $\{\pm 1, \pm 2\}$, and the factors of the leading coefficient, 3, are $\{\pm 1, \pm 3\}$. The possible rational zeros are $\left\{\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 2\right\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrrr} -2 & 3 & 5 & 1 & 5 & -2 \\ & & -6 & 2 & -6 & 2 \\ \hline & 3 & -1 & 3 & -1 & 0 \end{array}$$

So, –2 is a rational zero.
Use synthetic division again to find another rational zero:

$$\begin{aligned} &3x^4 + 5x^3 + x^2 + 5x - 2 \\ &\qquad = (x+2)(3x^3 - x^2 + 3x - 1) \end{aligned}$$

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & 3 & -1 \\ & & 1 & 0 & 1 \\ \hline & 3 & 0 & 3 & 0 \end{array}$$

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$$\begin{aligned}3x^4+5x^3+x^2+5x-2 &= (x+2)\left(x-\frac{1}{3}\right)(3x^2+3)\\ &= 3(x+2)\left(x-\frac{1}{3}\right)(x^2+1) \Rightarrow\end{aligned}$$

the rational zeros are $\left\{-2, \frac{1}{3}\right\}$.

25. The factors of the constant term, 1, are $\{\pm 1\}$, and the factors of the leading coefficient, 1, are $\{\pm 1\}$. The possible rational zeros are $\{\pm 1\}$. Use synthetic division to find one rational root:

$$\begin{array}{r|rrrrr} \underline{1} & 1 & -2 & 10 & -1 & 1 \\ & & 1 & -1 & 9 & 8 \\ \hline & 1 & -1 & 9 & 8 & 9 \end{array}$$

The remainder is 9 so, 1 is not a rational zero. Try −1:

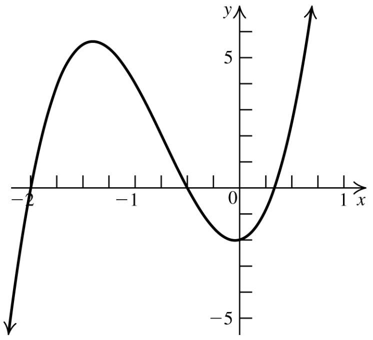
$$\begin{array}{r|rrrrr} \underline{-1} & 1 & -2 & 10 & -1 & 1 \\ & & -1 & 3 & -13 & 14 \\ \hline & 1 & -3 & 13 & -14 & 15 \end{array}$$

The remainder is 15 so, −1 is not a rational zero. Therefore, there are no rational zeros.

26. The factors of the constant term, 2, are $\{\pm 1, \pm 2\}$, and the factors of the leading coefficient, 1, are $\{\pm 1\}$.The possible rational zeros are $\{\pm 1, \pm 2\}$. Testing each value $(f(-2),\ f(-1), f(1), f(2))$ shows that none cause the value of the function to equal zero, so there are no rational zeros.

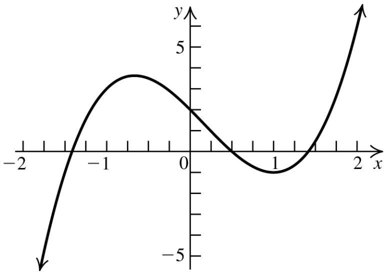
27. $f(x) = 6x^3 + 13x^2 + x - 2$

Zeros: $\left\{-2, -\frac{1}{2}, \frac{1}{3}\right\}$



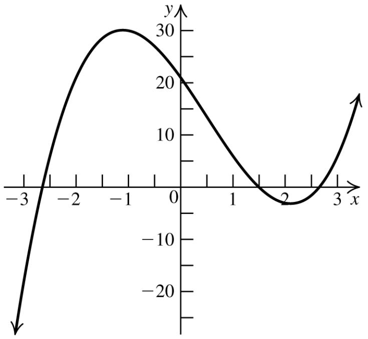
28. $f(x) = 2x^3 - x^2 - 4x + 2$

Zeros: $\left\{\pm\sqrt{2}, \frac{1}{2}\right\}$



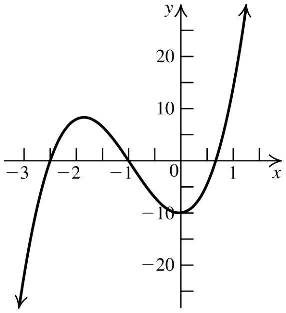
29. $f(x) = 2x^3 - 3x^2 - 14x + 21$

Zeros: $\left\{\pm\sqrt{7}, \frac{3}{2}\right\}$



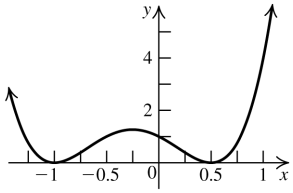
30. $f(x) = 6x^3 + 17x^2 + x - 10$

Zeros: $\left\{-\frac{5}{2}, -1, \frac{2}{3}\right\}$



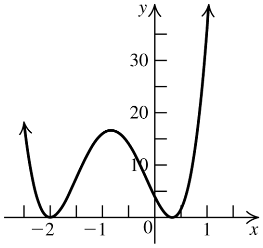
31. $f(x) = 4x^4 + 4x^3 - 3x^2 - 2x + 1$

Zeros: $\left\{-1, \frac{1}{2}\right\}$



32. $f(x) = 9x^4 + 30x^3 + 13x^2 - 20x + 4$

Zeros: $\left\{-2, \dfrac{1}{3}\right\}$



33. $f(x) = 5x^3 - 2x^2 - 3x + 4;$
$f(-x) = 5(-x)^3 - 2(-x)^2 - 3(-x) + 4$
$= -5x^3 - 2x^2 + 3x + 4$
There are two sign changes in $f(x)$, so there are either 2 or 0 positive zeros. There is one sign change in $f(-x)$, so there is one negative zero.

34. $g(x) = 3x^3 + x^2 - 9x - 3;$
$g(-x) = 3(-x)^3 + (-x)^2 - 9(-x) - 3$
$= -3x^3 + x^2 + 9x - 3$
There is one sign change in $g(x)$, so there is one positive zero. There are two sign changes in $g(-x)$, so there are either 2 or 0 negative zeros.

35. $f(x) = 2x^3 + 5x^2 - x + 2;$
$f(-x) = 2(-x)^3 + 5(-x)^2 - (-x) + 2$
$= -2x^3 + 5x^2 + x + 2$
There are two sign changes in $f(x)$, so there are either 2 or 0 positive zeros. There is one sign change in $f(-x)$, so there is one negative zero.

36. $g(x) = 3x^4 + 8x^3 - 5x^2 + 2x - 3;$
$g(-x) = 3(-x)^4 + 8(-x)^3 - 5(-x)^2 + 2(-x) - 3$
$= 3x^4 - 8x^3 - 5x^2 - 2x - 3$
There are three sign changes in $g(x)$, so there are either 3 or 1 positive zeros. There is one sign change in $g(-x)$, so there is 1 negative zero.

37. $h(x) = 2x^5 - 5x^3 + 3x^2 + 2x - 1$
$h(-x) = 2(-x)^5 - 5(-x)^3 + 3(-x)^2 + 2(-x) - 1$
$= -2x^5 + 5x^3 + 3x^2 - 2x - 1$
There are three sign changes in $h(x)$, so there are either 1 or 3 positive zeros. There are two sign changes in $h(-x)$, so there are either 2 or 0 negative zeros.

38. $F(x) = 5x^6 - 7x^4 + 2x^3 - 1;$
$F(-x) = 5(-x)^6 - 7(-x)^4 + 2(-x)^3 - 1$
$= 5x^6 - 7x^4 - 2x^3 - 1$
There are three sign changes in $F(x)$, so there are either 3 or 1 positive zeros. There is one sign change in $F(-x)$, so there is 1 negative zero.

39. $G(x) = -3x^4 - 4x^3 + 5x^2 - 3x + 7;$
$G(-x) = -3(-x)^4 - 4(-x)^3 + 5(-x)^2$
$- 3(-x) + 7$
$= -3x^4 + 4x^3 + 5x^2 + 3x + 7$
There are three sign changes in $G(x)$, so there are either 3 or 1 positive zeros. There is one sign change in $G(-x)$, so there is 1 negative zero.

40. $H(x) = -5x^5 + 3x^3 - 2x^2 - 7x + 4;$
$H(-x) = -5(-x)^5 + 3(-x)^3 - 2(-x)^2$
$-7(-x) + 4$
$= 5x^5 - 3x^3 - 2x^2 + 7x + 4$
There are three sign changes in $H(x)$, so there are either 3 or 1 positive zeros. There are two sign changes in $H(-x)$, so there are 0 or 2 negative zeros.

41. $f(x) = x^4 + 2x^2 + 4$
$f(-x) = (-x)^4 + 2(-x)^2 + 4 = x^4 + 2x^2 + 4$
There are no sign changes in $f(x)$, nor are there sign changes in $f(-x)$. Therefore, there are no positive zeros and no negative zeros.

42. $f(x) = 3x^4 + 5x^2 + 6$
$f(-x) = 3(-x)^4 + 5(-x)^2 + 6 = 3x^4 + 5x^2 + 6$
There are no sign changes in $f(x)$, nor are there sign changes in $f(-x)$. Therefore, there are no positive zeros and no negative zeros.

43. $g(x) = 2x^5 + x^3 + 3x$
$g(-x) = 2(-x)^5 + (-x)^3 + 3(-x)$
$= -2x^5 - x^3 - 3x$
There are no sign changes in $g(x)$, nor are there sign changes in $g(-x)$. Therefore, there are no positive zeros and no negative zeros.

44. $g(x) = 2x^5 + 4x^3 + 5x$

$$g(-x) = 2(-x)^5 + 4(-x)^3 + 5(-x)$$
$$= -2x^5 - 4x^3 - 5x$$

There are no sign changes in $g(x)$, nor are there sign changes in $g(-x)$. Therefore, there are no positive zeros and no negative zeros.

45. $h(x) = -x^5 - 2x^3 + 4$

$$h(-x) = -(-x)^5 - 2(-x)^3 + 4$$
$$= x^5 + x^3 + 4$$

There is one sign change in $h(x)$, and no sign changes in $g(-x)$. Therefore, there is one positive zero and there are no negative zeros.

46. $h(x) = 2x^5 + 3x^3 + 1$

$$h(-x) = 2(-x)^5 + 3(-x)^3 + 1$$
$$= -2x^5 - 3x^3 + 1$$

There are no sign changes in $h(x)$, and one sign change in $g(-x)$. Therefore, there are no positive zeros and there is one negative zero.

47. The possible rational zeros are $\left\{\pm\frac{1}{3}, \pm1, \pm3\right\}$.

There are three sign changes in $f(x)$, so there are either 3 or 1 positive zeros. There are no sign changes in $f(-x)$, so there are no negative zeros. Using synthetic division with $k = 1, 2, 3, \ldots$ and $k = -1, -2, -3, \ldots$ gives an upper bound of 1 and a lower bound of -1.

$$\begin{array}{r|rrrr} 1 & 3 & -1 & 9 & -3 \\ & & 3 & 2 & 11 \\ \hline & 3 & 2 & 11 & 8 \end{array} \qquad \begin{array}{r|rrrr} -1 & 3 & -1 & 9 & -3 \\ & & -3 & 4 & -13 \\ \hline & 3 & -4 & 13 & -16 \end{array}$$

48. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm3, \pm\frac{7}{2}, \pm7, \pm\frac{21}{2} \pm 21\right\}$.

There are two sign changes in $g(x)$, so there are either 2 or 0 positive zeros. There is one sign change in $g(-x)$, so there is one negative zero. Using synthetic division with $k = 1, 2, 3, \ldots$ and $k = -1, -2, -3, \ldots$ gives an upper bound of 4 and a lower bound of -3.

$$\begin{array}{r|rrrr} 4 & 2 & -3 & -14 & 21 \\ & & 8 & 20 & 24 \\ \hline & 2 & 5 & 6 & 45 \end{array} \qquad \begin{array}{r|rrrr} -3 & 2 & -3 & -14 & 21 \\ & & -6 & 27 & -39 \\ \hline & 2 & -9 & 13 & -18 \end{array}$$

49. The possible rational zeros are $\left\{\pm\frac{1}{3}, \pm1, \pm\frac{7}{3}, \pm7\right\}$. There are no sign changes in $F(x)$, so there are no positive zeros. There are three sign changes in $F(-x)$, so there are either 3 or 1 negative zeros. Using synthetic division with $k = 1, 2, 3, \ldots$ and $k = -1, -2, -3, \ldots$ gives an upper bound of 1 and a lower bound of -3.

$$\begin{array}{r|rrrr} 1 & 3 & 2 & 5 & 7 \\ & & 3 & 5 & 10 \\ \hline & 3 & 5 & 10 & 17 \end{array} \qquad \begin{array}{r|rrrr} -3 & 3 & 2 & 5 & 7 \\ & & -9 & 21 & -75 \\ \hline & 3 & -7 & 25 & -68 \end{array}$$

50. The possible rational zeros are $\{\pm1, \pm2, \pm4\}$. There is one sign change in $G(x)$, so there is one positive zero. There are two sign changes in $G(-x)$, so there are either 2 or 0 negative zeros. Using synthetic division with $k = 1, 2, 3, \ldots$ and $k = -1, -2, -3, \ldots$ gives an upper bound of 1 and a lower bound of -4.

$$\begin{array}{r|rrrr} 1 & 1 & 3 & 1 & -4 \\ & & 1 & 4 & 5 \\ \hline & 1 & 4 & 5 & 1 \end{array} \qquad \begin{array}{r|rrrr} -4 & 1 & 3 & 1 & -4 \\ & & -4 & 4 & -20 \\ \hline & 1 & -1 & 5 & -4 \end{array}$$

51. The possible rational zeros are $\{\pm1, \pm31\}$. There are two sign changes in $h(x)$, so there are either 2 or 0 positive zeros. There are two sign changes in $h(-x)$, so there are either 2 or 0 negative zeros. Using synthetic division with $k = 1, 2, 3, \ldots$ and $k = -1, -2, -3, \ldots$ gives an upper bound of 31 and a lower bound of -31.

$$\begin{array}{r|rrrrr} 31 & 1 & 3 & -15 & -9 & 31 \\ & & 31 & 1054 & 32{,}209 & 998{,}200 \\ \hline & 1 & 34 & 1039 & 32{,}200 & 998{,}231 \end{array}$$

$$\begin{array}{r|rrrrr} -31 & 1 & 3 & -15 & -9 & 31 \\ & & -31 & 868 & -26{,}443 & 820{,}012 \\ \hline & 1 & -28 & 853 & -26{,}452 & 820{,}043 \end{array}$$

52. The possible rational zeros are $\left\{\pm\frac{1}{3}, \pm1, \pm\frac{13}{3}, \pm13\right\}$. There are three sign changes in $H(x)$, so there are either 3 or 1 positive zeros. There is one sign change in $H(-x)$, so there is one negative zero. Using synthetic division with $k = 1, 2, 3, \ldots$ and $k = -1, -2, -3, \ldots$ gives an upper bound of 13 and a lower bound of -1.

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$$\begin{array}{r|rrrrr} \underline{13} & 3 & -20 & 28 & 19 & -13 \\ & & 39 & 247 & 3575 & 46{,}722 \\ \hline & 3 & 19 & 275 & 3594 & 46{,}709 \end{array}$$

$$\begin{array}{r|rrrrr} \underline{-1} & 3 & -20 & 28 & 19 & -13 \\ & & -3 & 23 & -51 & 32 \\ \hline & 3 & -23 & 51 & -32 & 19 \end{array}$$

53. The possible rational zeros are $\left\{\pm\frac{1}{6},\pm\frac{1}{3},\pm\frac{1}{2},\pm1,\pm\frac{7}{6},\pm\frac{7}{3},\pm\frac{7}{2},\pm7\right\}.$

There are two sign changes in $f(x)$, so there are either 2 or 0 positive zeros. There are two sign changes in $f(-x)$, so there are either 2 or 0 negative zeros. Using synthetic division with $k = 1, 2, 3, \ldots$ and $k = -1, -2, -3, \ldots$ gives an upper bound of 4 and a lower bound of -4.

$$\begin{array}{r|rrrrr} \underline{4} & 6 & 1 & -43 & -7 & 7 \\ & & 24 & 100 & 228 & 884 \\ \hline & 6 & 25 & 57 & 221 & 891 \end{array}$$

$$\begin{array}{r|rrrrr} \underline{-4} & 6 & 1 & -43 & -7 & 7 \\ & & -24 & 92 & -196 & 812 \\ \hline & 6 & -23 & 49 & -203 & 819 \end{array}$$

54. The possible rational zeros are $\left\{\pm\frac{1}{6},\pm\frac{1}{3},\pm\frac{1}{2},\pm\frac{5}{6},\pm1,\pm\frac{5}{3},\pm\frac{5}{2},\pm5\right\}.$

There is one sign change in $g(x)$, so there is one positive zero. There are three sign changes in $g(-x)$, so there are either 3 or 1 negative zeros. Using synthetic division with $k = 1, 2, 3, \ldots$ and $k = -1, -2, -3, \ldots$ gives an upper bound of 1 and a lower bound of -5

$$\begin{array}{r|rrrrr} \underline{1} & 6 & 23 & 25 & -9 & -5 \\ & & 6 & 29 & 54 & 45 \\ \hline & 6 & 29 & 54 & 45 & 40 \end{array}$$

$$\begin{array}{r|rrrrr} \underline{-5} & 6 & 23 & 25 & -9 & -5 \\ & & -30 & 35 & -300 & 1545 \\ \hline & 6 & -7 & 60 & -309 & 1540 \end{array}$$

In exercises 55−74, first check to see the number of possible positive zeros and the number of possible negative zeros.

55. $f(x) = x^3 + 5x^2 - 8x + 2$

The function has degree 3, so there are three zero, zero or two possible positive zeros; one possible negative zero.

The possible rational zeros are $\{\pm1, \pm2\}$.

$$\begin{array}{r|rrrr} \underline{1} & 1 & 5 & -8 & 2 \\ & & 1 & 6 & -2 \\ \hline & 1 & 6 & -2 & 0 \end{array}$$

So, 1 is a zero. Now solve the depressed equation $x^2 + 6x - 2 = 0.$

$$x = \frac{-6 \pm \sqrt{36 - 4(1)(-2)}}{2(1)} = \frac{-6 \pm \sqrt{44}}{2}$$
$$= -3 \pm \sqrt{11}$$

The solution set is $\left\{1, -3 \pm \sqrt{11}\right\}$.

56. $f(x) = x^3 - 7x^2 - 5x + 3$

The function has degree 3, so there are three zeros, zero or two possible positive zeros; one possible negative zero.

The possible rational zeros are $\{\pm1, \pm3\}$.

$$\begin{array}{r|rrrr} \underline{-1} & 1 & -7 & -5 & 3 \\ & & -1 & 8 & -3 \\ \hline & 1 & -8 & 3 & 0 \end{array}$$

So, −1 is a zero. Now solve the depressed equation $x^2 - 8x + 3 = 0.$

$$x = \frac{8 \pm \sqrt{64 - 4(1)(3)}}{2(1)} = \frac{8 \pm \sqrt{52}}{2}$$
$$= 4 \pm \sqrt{13}$$

The solution set is $\left\{-1, 4 \pm \sqrt{13}\right\}$.

57. $f(x) = 2x^3 - x^2 - 6x + 3$

The function has degree 3, so there are three zeros, zero or two possible positive zeros; one possible negative zero. The possible rational zeros are $\left\{\pm1, \pm3, \pm\frac{1}{2}, \pm\frac{3}{2}\right\}.$

$$\begin{array}{r|rrrr} \underline{\frac{1}{2}} & 2 & -1 & -6 & 3 \\ & & 1 & 0 & -3 \\ \hline & 2 & 0 & -6 & 0 \end{array}$$

So, $1/2$ is a zero. Now solve the depressed equation $2x^2 - 6 = 0.$

$$2x^2 - 6 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

Solution set: $\left\{-\sqrt{3}, \frac{1}{2}, \sqrt{3}\right\}$

58. $f(x)=3x^3+2x^2-15x-10$

The function has degree 3, so there are three zeros, one possible positive zero; zero or two possible negative zeros.
The possible rational zeros are
$\left\{\pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}\right\}.$

$$\begin{array}{r|rrrr} -\frac{2}{3} & 3 & 2 & -15 & -10 \\ & & -2 & 0 & 10 \\ \hline & 3 & 0 & -15 & 0 \end{array}$$

So, $-2/3$ is a zero. Now solve the depressed equation $3x^2-15=0.$

$3x^2-15=0 \Rightarrow x^2=5 \Rightarrow x=\pm\sqrt{5}$

Solution set: $\left\{-\sqrt{5}, -\frac{2}{3}, \sqrt{5}\right\}$

59. $f(x)=2x^3-9x^2+6x-1$

The function has degree 3, so there are three zeros, one or three possible positive zeros; no possible negative zeros.

The possible rational zeros are $\left\{\pm 1, \pm\frac{1}{2}\right\}.$

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -9 & 6 & -1 \\ & & 1 & -4 & 1 \\ \hline & 2 & -8 & 2 & 0 \end{array}$$

So, $1/2$ is a zero. Now solve the depressed equation $2x^2-8x+2=0$

$$x=\frac{-(-8)\pm\sqrt{(-8)^2-4(2)(2)}}{2(2)}=\frac{8\pm\sqrt{48}}{4}$$
$$=2\pm\sqrt{3}$$

Solution set: $\left\{2-\sqrt{3}, \frac{1}{2}, 2+\sqrt{3}\right\}.$

60. $f(x)=2x^3-3x^2-4x-1$

The function has degree 3, so there are three zeros, one possible positive zero; zero or two possible negative zeros.

The possible rational zeros are $\left\{\pm 1, \pm\frac{1}{2}\right\}.$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -3 & -4 & -1 \\ & & -1 & 2 & 1 \\ \hline & 2 & -4 & -2 & 0 \end{array}$$

So, $-1/2$ is a zero. Now solve the depressed equation $2x^2-4x-2=0$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(-2)}}{2(2)}=\frac{4\pm\sqrt{32}}{4}$$
$$=1\pm\sqrt{2}$$

Solution set: $\left\{1-\sqrt{2}, -\frac{1}{2}, 1+\sqrt{2}\right\}.$

61. $f(x)=x^4+x^3-5x^2-3x+6$

The function has degree 4, so there are four zeros, zero or two possible positive zeros; zero or two possible negative zeros.
The possible rational zeros are
$\{\pm 1, \pm 2, \pm 3, \pm 6\}.$

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & -5 & -3 & 6 \\ & & 1 & 2 & -3 & -6 \\ \hline & 1 & 2 & -3 & -6 & 0 \end{array}$$

So, 1 is a zero. Now find a zero of the depressed equation $x^3+2x^2-3x-6=0.$

$$\begin{array}{r|rrrr} -2 & 1 & 2 & -3 & -6 \\ & & -2 & 0 & 6 \\ \hline & 1 & 0 & -3 & 0 \end{array}$$

So, −2 is a zero. Now solve the depressed equation $x^2-3=0.$

$x^2-3=0 \Rightarrow x^2=3 \Rightarrow x=\pm\sqrt{3}$

Solution set: $\left\{-2, -\sqrt{3}, 1, \sqrt{3}\right\}$

62. $f(x)=x^4-2x^3-5x^2+4x+6$

The function has degree 4, so there are four zeros, zero or two possible positive real zeros; zero or two possible negative real zeros.
The possible rational zeros are
$\{\pm 1, \pm 2, \pm 3, \pm 6\}.$

$$\begin{array}{r|rrrrr} 3 & 1 & -2 & -5 & 4 & 6 \\ & & 3 & 3 & -6 & -6 \\ \hline & 1 & 1 & -2 & -2 & 0 \end{array}$$

So, 3 is a zero. Now find a zero of the depressed equation $x^3+x^2-2x-2=0.$

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -2 & -2 \\ & & -1 & 0 & 2 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

So, −1 is a zero. Now solve the depressed equation $x^2-2=0.$

$x^2-2=0 \Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$

Solution set: $\left\{-\sqrt{2}, -1, \sqrt{2}, 3\right\}$

63. $f(x) = x^4 - 3x^3 + 3x - 1$

The function has degree 4, so there are four zeros, one or three possible positive real zeros; one possible negative real zero. The possible rational zeros are $\{\pm 1\}$.

$$\begin{array}{r|rrrrr} 1 & 1 & -3 & 0 & 3 & -1 \\ & & 1 & -2 & -2 & 1 \\ \hline & 1 & -2 & -2 & 1 & 0 \end{array}$$

So, 1 is a zero. Now find a zero of the depressed equation $g(x) = x^3 - 2x^2 - 2x + 1$.

$$\begin{array}{r|rrrr} -1 & 1 & -2 & -2 & 1 \\ & & -1 & 3 & -1 \\ \hline & 1 & -3 & 1 & 0 \end{array}$$

So, -1 is a zero. Now solve the depressed equation $x^2 - 3x + 1 = 0$

$$x = \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}$$

The solution set is $\left\{\pm 1, \frac{3 \pm \sqrt{5}}{2}\right\}$.

64. $f(x) = x^4 - 6x^3 - 7x^2 + 54x - 18$

The function has degree 4, so there are four zeros. One or three possible positive real zeros; one possible negative real zero. The possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18\}$.

$$\begin{array}{r|rrrrr} 3 & 1 & -6 & -7 & 54 & -18 \\ & & 3 & -9 & -48 & 18 \\ \hline & 1 & -3 & -16 & 6 & 0 \end{array}$$

So, 3 is a zero. Now find a zero of the depressed equation

$g(x) = x^3 - 3x^2 - 16x + 6$.

$$\begin{array}{r|rrrr} -3 & 1 & -3 & -16 & 6 \\ & & -3 & 18 & -6 \\ \hline & 1 & -6 & 2 & 0 \end{array}$$

So, -3 is a zero. Now solve the depressed equation $x^2 - 6x + 2 = 0$

$$x = \frac{6 \pm \sqrt{36 - 4(1)(2)}}{2(1)} = \frac{6 \pm \sqrt{28}}{2} = 3 \pm \sqrt{7}$$

The solution set is $\{\pm 3, 3 \pm \sqrt{7}\}$.

65. $f(x) = 2x^4 - 5x^3 - 4x^2 + 15x - 6$

The function has degree 4, so there are four zeros, one or three possible positive real zeros; one possible negative real zero. The possible rational zeros are

$\left\{\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}\right\}$.

$$\begin{array}{r|rrrrr} 2 & 2 & -5 & -4 & 15 & -6 \\ & & 4 & -2 & -12 & 6 \\ \hline & 2 & -1 & -6 & 3 & 0 \end{array}$$

So, 2 is a zero. Now find a zero of the depressed equation $g(x) = 2x^3 - x^2 - 6x + 3$.

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & -6 & 3 \\ & & 1 & 0 & -3 \\ \hline & 2 & 0 & -6 & 0 \end{array}$$

So, $1/2$ is a zero. Now solve the depressed equation $2x^2 - 6 = 0$.

$2x^2 - 6 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

The solution set is $\left\{-\sqrt{3}, \frac{1}{2}, \sqrt{3}, 2\right\}$.

66. $f(x) = 3x^4 - 8x^3 - 18x^2 + 40x + 15$

The function has degree 4, so there are four zeros, zero or two possible positive real zeros; zero or two possible negative real zeros. The possible rational zeros are

$\left\{\pm 1, \pm 3, \pm 5, \pm 15, \pm\frac{1}{3}, \pm\frac{5}{3}\right\}$.

$$\begin{array}{r|rrrrr} 3 & 3 & -8 & -18 & 40 & 15 \\ & & 9 & 3 & -45 & -15 \\ \hline & 3 & 1 & -15 & -5 & 0 \end{array}$$

So, 3 is a zero. Now find a zero of the depressed equation $g(x) = 3x^3 + x^2 - 15x - 5$.

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & 1 & -15 & -5 \\ & & -1 & 0 & 5 \\ \hline & 3 & 0 & -15 & 0 \end{array}$$

So, $-1/3$ is a zero. Now solve the depressed equation $3x^2 - 15 = 0$.

$3x^2 - 15 = 0 \Rightarrow x^2 = 5 \Rightarrow x = \pm\sqrt{5}$

The solution set is $\left\{-\sqrt{5}, -\frac{1}{3}, \sqrt{5}, 3\right\}$.

67. $f(x) = 6x^4 - x^3 - 13x^2 + 2x + 2$

The function has degree 4, so there are four zeros, zero or two possible positive real zeros; zero or two possible negative real zeros. The possible rational zeros are

$\left\{\pm 1, \pm 3, \pm\frac{1}{6}, \pm\frac{1}{3}\right\}.$

$$\begin{array}{r|rrrrr} \frac{1}{2} & 6 & -1 & -13 & 2 & 2 \\ & & 3 & 1 & -6 & -2 \\ \hline & 6 & 2 & -12 & -4 & 0 \end{array}$$

So, $1/2$ is a zero. Now find a zero of the depressed equation

$g(x) = 6x^3 + 2x^2 - 12x - 4.$

$$\begin{array}{r|rrrr} -\frac{1}{3} & 6 & 2 & -12 & -4 \\ & & -2 & 0 & 4 \\ \hline & 6 & 0 & -12 & 0 \end{array}$$

So, $-1/3$ is a zero. Now solve the depressed equation $6x^2 - 12 = 0.$

$6x^2 - 12 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

The solution set is $\left\{-\sqrt{2}, -\frac{1}{3}, \frac{1}{2}, \sqrt{2}\right\}.$

68. $f(x) = 6x^4 + x^3 - 19x^2 - 3x + 3$

The function has degree 4, so there are four zeros, zero or two possible positive real zeros; zero or two possible negative real zeros. The possible rational zeros are

$\left\{\pm 1, \pm 3, \pm\frac{1}{6}, \pm\frac{1}{3}, \pm\frac{1}{2}\right\}.$

$$\begin{array}{r|rrrrr} \frac{1}{3} & 6 & 1 & -19 & -3 & 3 \\ & & 2 & 1 & -6 & -3 \\ \hline & 6 & 3 & -18 & -9 & 0 \end{array}$$

So, $1/3$ is a zero. Now find a zero of the depressed equation

$g(x) = 6x^3 + 3x^2 - 18x - 9.$

$$\begin{array}{r|rrrr} -\frac{1}{2} & 6 & 3 & -18 & -9 \\ & & -3 & 0 & 9 \\ \hline & 6 & 0 & -18 & 0 \end{array}$$

So, $-1/2$ is a zero. Now solve the depressed equation $6x^2 - 18 = 0.$

$6x^2 - 18 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

The solution set is $\left\{-\sqrt{3}, -\frac{1}{2}, \frac{1}{3}, \sqrt{3}\right\}.$

69. $f(x) = x^5 - 2x^4 - 4x^3 + 8x^2 + 3x - 6$

The function has degree 5, so there are five zeros. There are three sign changes, so there are either 1, 3, or 5 positive real zeros. There are two sign changes in $f(-x)$, so there are zero or two negative real zeros. The possible rational zeros are $\{\pm 1, \pm 3, \pm 6\}.$ Using synthetic division to test the positive values, we find that one zero is 1:

$$\begin{array}{r|rrrrrr} 1 & 1 & -2 & -4 & 8 & 3 & -6 \\ & & 1 & -1 & -5 & 3 & 6 \\ \hline & 1 & -1 & -5 & 3 & 6 & 0 \end{array}$$

The zeros of the depressed function $x^4 - x^3 - 5x^2 + 3x + 6$ are also zeros of *f*. Use synthetic division again to find the next zero, 2:

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & -5 & 3 & 6 \\ & & 2 & 2 & -6 & -6 \\ \hline & 1 & 1 & -3 & -3 & 0 \end{array}$$

$$x^5 - 2x^4 - 4x^3 + 8x^2 + 3x - 6 \\ = (x-1)(x-2)\left(x^3 + x^2 - 3x - 3\right)$$

Now find a zero of the depressed function $x^3 + x^2 - 3x - 3.$ Use synthetic division again to find the next zero, −1:

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -3 & -3 \\ & & -1 & 0 & 3 \\ \hline & 1 & 0 & -3 & 0 \end{array}$$

$$x^5 - 2x^4 - x^3 + 8x^2 - 10x + 4 \\ = (x-1)(x-2)(x+1)\left(x^2 - 3\right)$$

Now solve the depressed equation: $x^2 - 3 = 0.$

$x^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

Solution set: $\left\{-\sqrt{3}, -1, 1, \sqrt{3}, 2\right\}$

70. $f(x) = x^5 + x^4 - 6x^3 - 6x^2 + 8x + 8$

The function has degree 5, so there are five zeros. There are two sign changes, so there are either 0 or 2 positive real zeros. There are three sign changes in $f(-x)$, so there are one or three negative real zeros. The possible rational zeros are $\{\pm 1, \pm 2, \pm 4, \pm 8\}.$

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Using synthetic division to test the positive values, we find that one zero is 2:

$$\begin{array}{r|rrrrrr} 2 & 1 & 1 & -6 & -6 & 8 & 8 \\ & & 2 & 6 & 0 & -12 & -8 \\ \hline & 1 & 3 & 0 & -6 & -4 & 0 \end{array}$$

The zeros of the depressed function $x^4 + 3x^3 - 6x - 4$ are also zeros of *f*. Use synthetic division again to find the next zero, −2:

$$\begin{array}{r|rrrrr} -2 & 1 & 3 & 0 & -6 & -4 \\ & & -2 & -2 & 4 & 4 \\ \hline & 1 & 1 & -2 & -2 & 0 \end{array}$$

$$x^5 + x^4 - 6x^3 - 6x^2 + 8x + 8 = (x-2)(x+2)\left(x^3 + x^2 - 2x - 2\right).$$

Now find a zero of the depressed function $x^3 + x^2 - 2x - 2.$ Use synthetic division again to find the next zero, −1:

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -2 & -2 \\ & & -1 & 0 & 2 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

$$x^5 + x^4 - 6x^3 - 6x^2 + 8x + 8 = (x-2)(x+2)(x+1)\left(x^2 - 2\right)$$

Now solve the depressed equation:

$x^2 - 2 = 0.$

$x^2 - 2 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Solution set: $\left\{-2, -\sqrt{2}, -1, 2, \sqrt{2}\right\}$

71. $f(x) = 2x^5 + x^4 - 11x^3 - x^2 + 15x - 6$

The function has degree 5, so there are five zeros. There are three sign changes, so there are either 1, 3, or 5 positive real zeros. There are two sign changes in $f(-x)$, so there are zero or two negative real zeros. The possible rational zeros are $\left\{\pm 1, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}\right\}.$

Using synthetic division to test the positive values, we find that one zero is 1:

$$\begin{array}{r|rrrrrr} 1 & 2 & 1 & -11 & -1 & 15 & -6 \\ & & 2 & 3 & -8 & -9 & 6 \\ \hline & 2 & 3 & -8 & -9 & 6 & 0 \end{array}$$

The zeros of the depressed function $2x^4 + 3x^3 - 8x^2 - 9x + 6$ are also zeros of *f*. Use synthetic division again to find the next zero, −2:

$$\begin{array}{r|rrrrr} -2 & 2 & 3 & -8 & -9 & 6 \\ & & -4 & 2 & 12 & -6 \\ \hline & 2 & -1 & -6 & 3 & 0 \end{array}$$

$$2x^5 + x^4 - 11x^3 - x^2 + 15x - 6 = (x-1)(x+2)\left(2x^3 - x^2 - 6x + 3\right)$$

Now find a zero of the depressed function $2x^3 - x^2 - 6x + 3.$ Use synthetic division again to find the next zero, $1/2$:

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -1 & -6 & 3 \\ & & 1 & 0 & -3 \\ \hline & 2 & 0 & -6 & 0 \end{array}$$

$$2x^5 + x^4 - 11x^3 - x^2 + 15x - 6 = (x-1)(x+2)\left(x - \frac{1}{2}\right)\left(2x^2 - 6\right)$$

Now solve the depressed equation:

$2x^2 - 6 = 0.$

$2x^2 - 6 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

Solution set: $\left\{-2, -\sqrt{3}, 1, \frac{1}{2}, \sqrt{3}\right\}$

72. $f(x) = 2x^5 - x^4 - 9x^3 + 10x + 4$

The function has degree 5, so there are five zeros. There are three sign changes, so there are either 1, 3, or 5 positive real zeros. There are two sign changes in $f(-x)$, so there are zero or two negative real zeros.

The possible rational zeros are $\left\{\pm 1, \pm 2, \pm 4, \pm\frac{1}{2}\right\}.$

Using synthetic division to test the positive values, we find that one zero is 2:

$$\begin{array}{r|rrrrrr} 2 & 2 & -1 & -9 & 0 & 10 & 4 \\ & & 4 & 6 & -6 & -12 & -4 \\ \hline & 2 & 3 & -3 & -6 & -2 & 0 \end{array}$$

The zeros of the depressed function $2x^4 + 3x^3 - 3x^2 - 6x - 2$ are also zeros of *f*. Use synthetic division again to find the next zero, −1.

$$\begin{array}{r|rrrrr} -1 & 2 & 3 & -3 & -6 & -2 \\ & & -2 & -1 & 4 & 2 \\ \hline & 2 & 1 & -4 & -2 & 0 \end{array}$$

$$2x^5 - x^4 - 9x^3 + 10x + 4 = (x-2)(x+1)\left(2x^3 + x^2 - 4x - 2\right).$$

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The zeros of the depressed function $2x^3+x^2-4x-2$ are also zeros of *P*. Use synthetic division again to find the next zero, $-1/2$:

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 1 & -4 & -2 \\ & & -1 & 0 & 2 \\ \hline & 2 & 0 & -4 & 0 \end{array}$$

$$2x^5-x^4-9x^3+10x+4$$
$$=(x-2)(x+1)\left(x+\frac{1}{2}\right)\left(2x^2-4\right)$$

Now solve the depressed equation:

$2x^2-4=0.$

$2x^2-4=0 \Rightarrow x^2=2 \Rightarrow x=\pm\sqrt{2}$

Solution set: $\left\{-\sqrt{2},\ -1,\ -\frac{1}{2},\ \sqrt{2},\ 2\right\}$

73. $f(x)=2x^5-13x^4+27x^3-17x^2-5x+6$

The function has degree 5, so there are five zeros. There are four sign changes, so there are either 0, 2, or 4 positive real zeros. There is one sign change in $f(-x)$, so there is one possible negative real zero. The possible rational zeros are

$\left\{\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm\frac{1}{2},\ \pm\frac{3}{2}\right\}.$

Using synthetic division to test the positive values, we find that one zero is 1:

$$\begin{array}{r|rrrrrr} 1 & 2 & -13 & 27 & -17 & -5 & 6 \\ & & 2 & -11 & 16 & -1 & -6 \\ \hline & 2 & -11 & 16 & -1 & -6 & 0 \end{array}$$

The zeros of the depressed function $2x^4-11x^3+16x^2-x-6$ are also zeros of *f*. Use synthetic division again to find the next zero, 1:

$$\begin{array}{r|rrrrr} 1 & 2 & -11 & 16 & -1 & -6 \\ & & 2 & -9 & 7 & 6 \\ \hline & 2 & -9 & 7 & 6 & 0 \end{array}$$

$$2x^5-13x^4+27x^3-17x^2-5x+6$$
$$=(x-1)(x-1)\left(2x^3-9x^2+7x+6\right)$$

The zeros of the depressed function $2x^3-9x^2+7x+6$ are also zeros of *P*. Use synthetic division again to find the next zero, 2.

$$\begin{array}{r|rrrr} 2 & 2 & -9 & 7 & 6 \\ \hline & & 4 & -10 & -6 \\ & 2 & -5 & -3 & 0 \end{array}$$

$$2x^5-13x^4+27x^3-17x^2-5x+6$$
$$=(x-1)(x-1)(x-2)\left(2x^2-5x-3\right)$$

Now solve the depressed equation:

$2x^2-5x-3=0.$

$2x^2-5x-3=0 \Rightarrow (2x+1)(x-3)=0 \Rightarrow$

$x=-\frac{1}{2},\ 3$

Solution set: $\left\{-\frac{1}{2},\ 1\text{ (multiplicity 2)},\ 2,\ 3\right\}$

74. $f(x)=3x^5+x^4-9x^3-3x^2+6x+2$

The function has degree 5, so there are five zeros. There are two sign changes, so there are either 0 or 2 possible positive real zeros. There are three sign changes in $f(-x)$, so there are 1 or 3 possible negative real zeros. The possible rational zeros are $\left\{\pm 1,\ \pm 2,\ \pm\frac{1}{3},\ \pm\frac{2}{3}\right\}$. Using synthetic division to test the positive values, we find that one zero is 1:

$$\begin{array}{r|rrrrrr} 1 & 3 & 1 & -9 & -3 & 6 & 2 \\ & & 3 & 4 & -5 & -8 & -2 \\ \hline & 3 & 4 & -5 & -8 & -2 & 0 \end{array}$$

The zeros of the depressed function $3x^4+4x^3-5x^2-8x-2$ are also zeros of *f*. Use synthetic division again to find the next zero, -1:

$$\begin{array}{r|rrrrr} -1 & 3 & 4 & -5 & -8 & -2 \\ & & -3 & -1 & 6 & 2 \\ \hline & 3 & 1 & -6 & -2 & 0 \end{array}$$

$$3x^5+x^4-9x^3-3x^2+6x+2$$
$$=(x-1)(x+1)\left(3x^3+x^2-6x-2\right).$$

The zeros of the depressed function $3x^3+x^2-6x-2$ are also zeros of *P*. Use synthetic division again to find the next zero, $-\frac{1}{3}$.

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & 1 & -6 & -2 \\ & & -1 & 0 & 2 \\ \hline & 3 & 0 & -6 & 0 \end{array}$$

$$3x^5+x^4-9x^3-3x^2+6x+2$$
$$=(x-1)(x+1)\left(x+\frac{1}{3}\right)\left(3x^2-6\right)$$

(*continued on next page*)

(*continued*)

Now solve the depressed equation:

$3x^2 - 6 = 0.$

$3x^2 - 6 = 0 \Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Solution set: $\left\{-\sqrt{2}, -1, -\frac{1}{3}, 1, \sqrt{2}\right\}$

3.4 Applying the Concepts

75. The length of the rectangle is $x^2 - 2x + 3$ and its width is $x - 2$. Its area is 306 square units, so we have $\left(x^2 - 2x + 3\right)\left(x - 2\right) = 306 \Rightarrow$

$x^3 - 4x^2 + 7x - 6 = 306 \Rightarrow$

$x^3 - 4x^2 + 7x - 312 = 0.$

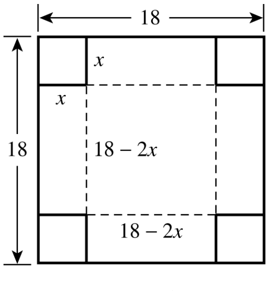
There are 3 sign changes in $f(x)$ and no sign changes in $f(-x)$, so there are 1 or 3 possible positive real zeros and no possible negative real zeros. The possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 13, \pm 24, \pm 26, \pm 39, \pm 52, \pm 78, \pm 104, \pm 156, \pm 312\}$

Using synthetic division, we find that 8 is a zero:

$$\begin{array}{r|rrrr} 8 & 1 & -4 & 7 & -312 \\ & & 8 & 32 & 312 \\ \hline & 1 & 4 & 39 & 0 \end{array}$$

Note that the discriminant of the depressed equation $x^2 + 4x + 39 = 0,\ 4^2 - 4(1)(39) < 0,$ so there are 2 complex solutions to the depressed equation. Therefore, $x = 8$. The width of the rectangle is $8 - 2 = 6$ units and the length of the rectangle is $8^2 - 2 \cdot 8 + 3 = 51$ units.

76. The length and width of the box are $18 - 2x$, and the height is x, so the volume is $x(18 - 2x)^2$.



$432 = x(18 - 2x)^2 \Rightarrow$

$432 = 4x^3 - 72x^2 + 324x \Rightarrow$

$4x^3 - 72x^2 + 324x - 432 = 0.$

The factors of the constant term, 432, are $\{\pm 1, \pm 2, \pm 4, \pm 6, \pm 8, \pm 12, \pm 16, \pm 18, \pm 24, \pm 27, \pm 36, \pm 54, \pm 108, \pm 216, \pm 432\}.$ The factors of the leading coefficient, 4, are $\{\pm 1, \pm 2, \pm 4\}$. There are many possible rational zeros, but the only ones that make sense for the problem are $\left\{\frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4, 6, 8\right\}$. Use synthetic division to find the zero:

$$\begin{array}{r|rrrr} 3 & 4 & -72 & 324 & -432 \\ & & 12 & -180 & 432 \\ \hline & 4 & -60 & 144 & 0 \end{array}$$

The corners should be 3 inches by 3 inches.

77. Use synthetic division to solve the equation $628 = 3x^3 - 6x^2 + 108x + 100 \Rightarrow$ $3x^3 - 6x^2 + 108x - 528 = 0.$ The factors of the constant term, −528, are $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 11, \pm 48, \pm 66, \pm 88, \pm 132, \pm 176, \pm 264, \pm 528\}.$ The factors of the leading coefficient, 3, are $\{\pm 1, \pm 3\}.$ Only the positive, whole number possibilities make sense for the problem, so the possible rational zeros are {1, 2, 3, 4, 6, 8, 11, 16, 22, 44, 48, 66, 88, 132, 176, 264, 528}.

$$\begin{array}{r|rrrr} 4 & 3 & -6 & 108 & -528 \\ & & 12 & 24 & 528 \\ \hline & 3 & 6 & 132 & 0 \end{array}$$

Thus, $x = 4$.

78. The demand function is $p(x) = 330 + 10x - x^2 \Rightarrow$ the revenue function is $330x + 10x^2 - x^3$.

Profit = revenue – cost =

$$910 = (330x + 10x^2 - x^3) \\ - (3x^3 - 6x^2 + 108x + 100) \Rightarrow$$

$910 = -4x^3 + 16x^2 + 222x - 100 \Rightarrow$

$0 = -4x^3 + 16x^2 + 222x - 1010$.

The factors of the constant term, –1010, are $\{\pm 1, \pm 2, \pm 5, \pm 10, \pm 101, \pm 202, \pm 505, \pm 1010\}.$

The factors of the leading coefficient, –4, are $\{\pm 1, \pm 2, \pm 4\}$. Find the zero using synthetic division:

$$\begin{array}{r|rrrr} 5 & -4 & 16 & 222 & -1010 \\ & & -20 & -20 & 1010 \\ \hline & -4 & -4 & 202 & 0 \end{array}$$

Thus, $x = 5$.

79. The cost function gives the result as a number of thousands, so set it equal to 125:

$x^3-15x^2+5x+50=125\Rightarrow$

$x^3-15x^2+5x-75=0$. The factors of the constant term, –75, are $\{\pm1,\pm3,\pm5,\pm15,$ $\pm25,\pm75\}$. The factors of the leading coefficient, 1, are $\{\pm1\}$. Only the positive solutions make sense for the problem, so the possible rational zeros are {1, 3, 5, 15, 25, 75}. Use synthetic division to find the zero:

$$\begin{array}{r|rrrr} 15 & 1 & -15 & 5 & -75 \\ & & 15 & 0 & 75 \\ \hline & 1 & 0 & 5 & 0 \end{array}$$

The total monthly cost is $125,000 when 1500 units are produced.

80. The demand function is $p(x)=-3x+3+\dfrac{74}{x}$, so the revenue function is $xp(x)=-3x^2+3x+74$. The break-even point occurs when revenue – cost = 0:

$(-3x^2+3x+74)-(x^3-15x^2+5x+50)=0\Rightarrow$

$-x^3+12x^2-2x+24=0\Rightarrow$

$(-x^3-2x)+(12x^2+24)=0\Rightarrow$

$-x(x^2+2)+12(x^2+2)=(12-x)(x^2+2)=0\Rightarrow$

$x=12$.

So 1200 printers must be sold to break even.

81. a. The country exported oil for 5 years, from 2007 to 2012.

b. There are three zeros, so the minimum degree of the polynomial is 3. The zeros are 0, 3, and 8, so the polynomial is of the form $p(x)=a(x-0)(x-3)(x-8)$. The graph goes through the point (9, –3), so we have

$-3=a(9-0)(9-3)(9-8)\Rightarrow -3=54a\Rightarrow$

$a=-\dfrac{1}{18}$

Thus, the equation is

$p(x)=y=-\dfrac{1}{18}x(x-3)(x-8)$.

c. $p(5)=-\dfrac{1}{18}(5)(5-3)(5-8)\approx1.7$

In 2009, the country exported about 1.7 million barrels of oil.

82. a. There are four zeros, so the minimum degree of the polynomial is 4. The zeros are 4, 6, 8, and 11, so the polynomial is of the form

$p(x)=a(x-4)(x-6)(x-8)(x-11)$.

The graph goes through the point (1, 350), so we have

$350=a(1-4)(1-6)(1-8)(1-11)\Rightarrow$

$350=1050a\Rightarrow a=\dfrac{1}{3}$

Thus, the equation is

$p(x)=y=\dfrac{1}{3}(x-4)(x-6)(x-8)(x-11)$.

b. $p(9)=\dfrac{1}{3}(9-4)(9-6)(9-8)(9-11)=-10$

In September, tourism was 10,000 below normal.

83. a. There are three zeros, so the minimum degree of the polynomial is 3. The zeros are 0, 6, and 8, so the polynomial is of the form $p(x)=a(x-0)(x-6)(x-8)$. The graph goes through the point (9, 0.5), so we have

$0.5=a(9-0)(9-6)(9-8)\Rightarrow$

$0.5=27a\Rightarrow a=\dfrac{1}{54}$

Thus, the equation is

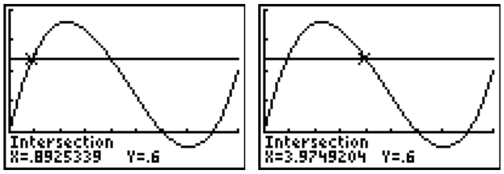
$p(x)=y=\dfrac{1}{54}x(x-6)(x-8)$.

b. $p(4)=\dfrac{1}{54}(4)(4-6)(4-8)\approx0.59259$

The profit in 2008 was about $592.59.

c. Set $Y_1=\dfrac{1}{54}x(x-6)(x-8)$ and $Y_2=0.6$.

Find the intersection of the two graphs.



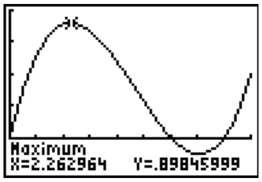
```
Intersection
X=.8925339    Y=.6
```

```
Intersection
X=3.9749204  Y=.6
```

[0, 9, 1] by [–0.25, 1, 0.25]

Ms. Sharpy realized a profit of $600 in 2004 and 2007.

d.



```
Maximum
X=2.262964    Y=.89845999
```

The maximum profit was approximately $898.46 in 2006.

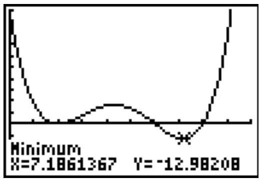
84. a. There are three turning points, so there are four zeros, 2 (multiplicity 2), 6, and 8. The polynomial is of the form $p(x)=a(x-2)(x-2)(x-6)(x-8)$. The graph goes through the point (0, 96), so we have $96=a(0-2)(0-2)(0-6)(0-8)\Rightarrow$ $96=192a\Rightarrow a=\frac{1}{2}$. Thus, the equation is $p(x)=y=\frac{1}{2}(x-2)^2(x-6)(x-8)$.

b. $p(3)=\frac{1}{2}(3-2)^2(3-6)(3-8)=7.5$

$p(7)=\frac{1}{2}(7-2)^2(7-6)(7-8)=-12.5$

In 2007, there was a surplus of \$7.5 million. In 2011, there was a deficit of \$12.5 million.

c.



Minimum
X=7.1861367 Y=-12.98208

The maximum deficit was approximately \$13 million in 2011.

3.4 Beyond the Basics

85. Let $x=\sqrt{3}\Rightarrow x^2=3\Rightarrow x^2-3=0$. The only possible rational zeros of this equation are ±1 and ±3. Because $\sqrt{3}$ is neither of these, it must be irrational.

86. $x=\sqrt[3]{4}\Rightarrow x^3=\left(\sqrt[3]{4}\right)^3\Rightarrow x^3-4=0.$
The only possible rational zeros of this equation are $\{\pm1,\pm2,\pm4\}$. Since $\sqrt[3]{4}$ is not in the set, it must be irrational.

87. $x=3-\sqrt{2}\Rightarrow 3-x=\sqrt{2}\Rightarrow 9-6x+x^2=2\Rightarrow$ $x^2-6x+7=0.$ The possible rational zeros of this equation are $\{\pm1,\pm7\}$. Since $3-\sqrt{2}$ is not in the set, it must be irrational.

88. $x=9^{2/3}\Rightarrow x^3=\left(9^{2/3}\right)^3\Rightarrow x^3-81=0.$ The possible rational zeros of this equation are $\{\pm1,\pm3,\pm9,\pm27,\ \pm81\}$. Since $9^{2/3}$ is not in the set, it must be irrational.

89. $y=(x-1)(x+1)(x-0)=x^3-x$

90. $y=(x-2)(x-1)(x+2)=x^3-x^2-4x+4$

91.
$$\begin{aligned}y&=\left(x+\frac{1}{2}\right)(x-2)\left(x-\frac{7}{3}\right)\\&=(2x+1)(x-2)(3x-7)\\&=6x^3-23x^2+15x+14\end{aligned}$$

92.
$$\begin{aligned}y&=\left(x+\frac{1}{3}\right)\left(x+\frac{1}{2}\right)\left(x-\frac{1}{6}\right)\\&=(3x+1)(2x+1)(6x-1)\\&=36x^3+24x^2+x-1\end{aligned}$$

93.
$$\begin{aligned}y&=\left(x-\left(1+\sqrt{2}\right)\right)\left(x-\left(1-\sqrt{2}\right)\right)(x-3)\\&=\left(x^2-2x-1\right)(x-3)=x^3-5x^2+5x+3\end{aligned}$$

94.
$$\begin{aligned}y&=\left(x-\left(3-\sqrt{5}\right)\right)\left(x-\left(3+\sqrt{5}\right)\right)(x-2)\\&=\left(x^2-6x+4\right)(x-2)=x^3-8x^2+16x-8\end{aligned}$$

95.
$$\begin{aligned}y&=\left(x-\left(1+\sqrt{3}\right)\right)\left(x-\left(1-\sqrt{3}\right)\right)\cdot\left(x-\left(3-\sqrt{2}\right)\right)\left(x-\left(3+\sqrt{2}\right)\right)=\left(x^2-2x-2\right)\left(x^2-6x+7\right)\\&=x^4-8x^3+17x^2-2x-14\end{aligned}$$

96.
$$\begin{aligned}y&=\left(x-\left(2-\sqrt{5}\right)\right)\left(x-\left(2+\sqrt{5}\right)\right)\cdot\left(x-\left(2+\sqrt{3}\right)\right)\left(x-\left(2-\sqrt{3}\right)\right)=\left(x^2-4x-1\right)\left(x^2-4x+1\right)\\&=x^4-8x^3+16x^2-1\end{aligned}$$

97. $y=\left(x-\left(\sqrt{2}+\sqrt{3}\right)\right)\left(x-\left(\sqrt{2}-\sqrt{3}\right)\right)=x^2-2x\sqrt{2}-1$

Since we must have integer coefficients, there must be additional zeros. Try $-\left(\sqrt{2}+\sqrt{3}\right)$:

$$y=\left(x-\left(\sqrt{2}+\sqrt{3}\right)\right)\left(x-\left(\sqrt{2}-\sqrt{3}\right)\right)\cdot\left(x+\left(\sqrt{2}+\sqrt{3}\right)\right)\left(x+\left(\sqrt{2}-\sqrt{3}\right)\right)=x^4-10x^2+1$$

98. $y=\left(x-\left(2+\left(\sqrt{3}+\sqrt{5}\right)\right)\right)\left(x-\left(2-\left(\sqrt{3}+\sqrt{5}\right)\right)\right)=x^2-4x-2\sqrt{15}-4$

Since we must have integer coefficients, there must be additional zeros. Try $2+\left(\sqrt{3}-\sqrt{5}\right)$:

$$y=\left(x-\left(2+\left(\sqrt{3}+\sqrt{5}\right)\right)\right)\left(x-\left(2-\left(\sqrt{3}+\sqrt{5}\right)\right)\right)\cdot\left(x-\left(2+\left(\sqrt{3}-\sqrt{5}\right)\right)\right)\cdot\left(x-\left(2-\left(\sqrt{3}-\sqrt{5}\right)\right)\right)$$
$$=x^4-8x^3+8x^2+32x-44$$

99. $x=3-\sqrt{2}+\sqrt{5}\Rightarrow x-3=-\sqrt{2}+\sqrt{5}\Rightarrow(x-3)^2=\left(-\sqrt{2}+\sqrt{5}\right)^2\Rightarrow x^2-6x+9=7-2\sqrt{10}\Rightarrow$
$x^2-6x+2=-2\sqrt{10}\Rightarrow\left(x^2-6x+2\right)^2=\left(-2\sqrt{10}\right)^2\Rightarrow x^4-12x^3+40x^2-24x+4=40\Rightarrow$
$y=x^4-12x^3+40x^2-24x-36$

100. $x=\sqrt{2}+\sqrt{3}+\sqrt{5}\Rightarrow x-\sqrt{5}=\sqrt{2}+\sqrt{3}\Rightarrow\left(x-\sqrt{5}\right)^2=\left(\sqrt{2}+\sqrt{3}\right)^2\Rightarrow x^2-2\sqrt{5}x+5=2+2\sqrt{6}+3\Rightarrow$
$x^2-2\sqrt{5}x=2\sqrt{6}\Rightarrow x^2=2\sqrt{5}x+2\sqrt{6}\Rightarrow\left(x^2\right)^2=\left(2\sqrt{5}x+2\sqrt{6}\right)^2\Rightarrow x^4=20x^2+8\sqrt{30}x+24\Rightarrow$
$x^4-20x^2-24=8\sqrt{30}x\Rightarrow\left(x^4-20x^2-24\right)^2=\left(8\sqrt{30}x\right)^2\Rightarrow x^8-40x^6+352x^4+960x^2+576=1920x^2\Rightarrow$
$y=x^8-40x^6+352x^4-960x^2+576$

101. i. Simplifying the fraction if necessary, we can assume that $\frac{p}{q}$ is in lowest terms.

Since $\frac{p}{q}$ is a zero of *F*, we have
$F\left(\frac{p}{q}\right)=0.$

ii. Substitute $\frac{p}{q}$ for *x* in the equation $F(x)=0.$

iii. Multiply the equation in (ii) by q^n.

iv. Subtract a_0q^n from both sides of the equation.

v. The left side of the equation in (iv) is
$a_np^n+a_{n-1}p^{n-1}q+\cdots+a_1pq^{n-1}=$
$p\left(a_np^{n-1}+a_{n-1}p^{n-2}q+\cdots+a_1q^{n-1}\right).$
Therefore *p* is a factor.

vi. $a=b\Leftrightarrow\frac{a}{p}=\frac{b}{p}.$

vii. Since *p* and *q* have no common prime factors, *p* must be a factor of a_0.

viii. Rearrange the terms of the equation in (iii).

ix. The left side of the equation in (viii) is
$a_{n-1}p^{n-1}q+\cdots+a_1pq^{n-1}+a_0q^n=$
$q\left(a_{n-1}p^{n-1}+\cdots+a_1pq^{n-2}+a_0q^{n-1}\right).$
Therefore *q* is a factor.

x. $a=b\Leftrightarrow\frac{a}{q}=\frac{b}{q}.$

xi. Since *p* and *q* have no common prime factors, *q* must be a factor of a_n.

102. $2x^4+2x^3+\frac{1}{2}x^2+2x-\frac{3}{2}=0\Rightarrow$
$4x^4+4x^3+x^2+4x-3=0.$ The factors of the constant term, –3, are $\{\pm1,\pm3\}$. The factors of the leading coefficient are $\{\pm1,\pm2,\pm4\}$. So, the possible rational zeros are
$\left\{\pm\frac{1}{4},\pm\frac{1}{2},\pm\frac{3}{4},\pm1,\pm\frac{3}{2},\pm3\right\}$. Use synthetic division to find one zero:

$$\begin{array}{r|rrrrr} \frac{1}{2} & 4 & 4 & 1 & 4 & -3 \\ & & 2 & 3 & 2 & 3 \\ \hline & 4 & 6 & 4 & 6 & 0 \end{array}$$

(*continued on next page*)

(*continued*)

So, $\frac{1}{2}$ is a rational zero and

$$\begin{aligned} &4x^4+4x^3+x^2+4x-3\\ &=\left(x-\frac{1}{2}\right)(4x^3+6x^2+4x+6)=\\ &=2\left(x-\frac{1}{2}\right)(2x^3+3x^2+2x+3)=\\ &=2\left(x-\frac{1}{2}\right)\left(x^2(2x+3)+1(2x+3)\right)=\\ &=2\left(x-\frac{1}{2}\right)\left(x^2+1\right)\left(2x+3\right)=0\Rightarrow \end{aligned}$$

$x=-\frac{3}{2}$ is another rational zero.

3.4 Critical Thinking/Discussion/Writing

103. a. False. The factors of the constant term, 3, are $\{\pm1,\pm3\}$ and the factors of the leading coefficient, 1, are $\{\pm1\}$. The possible rational zeros are $\{\pm1,\pm3\}$.

b. False. The factors of the constant term, 25, are $\{\pm1,\pm5,\pm25\}$ and the factors of the leading coefficient, 2, are $\{\pm1,\pm2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2},\pm1,\pm\frac{5}{2},\pm5,\pm\frac{25}{2},\pm25\right\}$.

104. Since $f(x)$ has integer coefficients, if $x=\sqrt{2}$ is a zero, then so is $x=-\sqrt{2}$. Then, two of the factors of $f(x)$ are $\left(x+\sqrt{2}\right)$ and $\left(x-\sqrt{2}\right)$. $\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)=x^2-2,$ so divide $f(x)$ by x^2-2 to find another factor.

$$\begin{array}{r} x^2-2x-2 \\ x^2+0x-2\overline{\smash{)}x^4-2x^3-4x^2+4x+4} \\ \underline{x^4+0x^3-2x^2} \\ -2x^3-2x^2+4x \\ \underline{-2x^3-0x^2+4x} \\ -2x^2+0x+4 \\ \underline{-2x^2+0x+4} \end{array}$$

Use the quadratic formula to solve $x^2-2x-2=0.$

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-2)}}{2(1)}=\frac{2\pm\sqrt{4+8}}{2}$$
$$=\frac{2\pm2\sqrt{3}}{2}=1\pm\sqrt{3}$$

The rational roots of the equation are $x\pm\sqrt{2}$ and $1\pm\sqrt{3}.$

3.4 Maintaining Skills

105. $(-2i)^5=(-2)^5(i)^5=(-2)^5(i)^4\,i=-32i$

106. $(x+i)(x-i)=x^2-i^2=x^2-(-1)=x^2+1$

107. $(2x+i)(-2x+i)=-4x^2+i^2=-4x^2-1$

108. $(x+3i)(x-3i)=x^2-9i^2=x^2+9$

109. $(5x+2i)(5x-2i)=25x^2-4i^2=25x^2+4$

110. $\begin{aligned}(2x+8i)(x-4i)&=2x^2-8ix+8ix-32i^2\\&=2x^2-32i^2=2x^2+32\end{aligned}$

---

111. $\begin{aligned}(x-1+2i)(x-1-2i)&=\left((x-1)+2i\right)\left((x-1)-2i\right)=(x-1)^2-4i^2\\&=x^2-2x+1+4=x^2-2x+5\end{aligned}$

112. $\begin{aligned}\left[x-(3-i)\right]\left[x-(3+i)\right]&=x^2-(3+i)x-(3-i)x+(3-i)(3+i)\\&=x^2-3x-ix-3x+ix+9-i^2=x^2-6x+9-(-1)=x^2-6x+10\end{aligned}$

113. $\begin{aligned}\left[x-(2+3i)\right]\left[x-(2-3i)\right]&=x^2-(2-3i)x-(2+3i)x+(2+3i)(2-3i)\\&=x^2-2x+3ix-2x-3ix+4-9i^2=x^2-4x+4+9=x^2-4x+13\end{aligned}$

114. $\begin{aligned}\left[x-\left(1+\sqrt{3}i\right)\right]\left[x-\left(1-\sqrt{3}i\right)\right]&=x^2-\left(1-\sqrt{3}i\right)x-\left(1+\sqrt{3}i\right)x+\left(1+\sqrt{3}i\right)\left(1-\sqrt{3}i\right)\\&=x^2-x-\sqrt{3}ix-x+\sqrt{3}ix+1-3i^2=x^2-2x+1+3=x^2-2x+4\end{aligned}$

3.5 The Complex Zeros of a Polynomial Function

3.5 Practice Problems

1. a. $$\begin{aligned} P(x) &= 3(x+2)(x-1)\left[x-(1+i)\right]\left[x-(1-i)\right] \\ &= 3(x+2)(x-1)(x-1-i)(x-1+i) \end{aligned}$$

b. $$\begin{aligned} P(x) &= 3(x+2)(x-1)\left[x-(1+i)\right]\left[x-(1-i)\right] \\ &= 3(x+2)(x-1)\left(x^2-2x+2\right) \\ &= 3(x+2)\left(x^3-3x^2+4x-2\right) \\ &= 3\left(x^4-x^3-2x^2+6x-4\right) \\ &= 3x^4-3x^3-6x^2+18x-12 \end{aligned}$$

2. Since 2 – 3*i* is a zero of multiplicity 2, so is 2 + 3*i*. Since *i* is a zero, so is –*i*. The eight zeros are –3, –3, 2 – 3*i*, 2 – 3*i*, 2 + 3*i*, 2 + 3*i*, *i*, and –*i*.

3. The function has degree four, so there are four zeros. Since one zero is $2i$, another zero is $-2i$. So $(x-2i)(x+2i) = x^2+4$ is a factor of $P(x)$. Now divide to find the other factors.

$$\begin{array}{rl}
 & x^2 - 3x + 2 \\ \hline
x^2 + 4 \big) & x^4 - 3x^3 + 6x^2 - 12x + 8 \\
 & \underline{x^4 \qquad + 4x^2} \\
 & -3x^3 + 2x^2 - 12x \\
 & \underline{-3x^3 \qquad - 12x} \\
 & 2x^2 \qquad + 8 \\
 & \underline{2x^2 \qquad + 8} \\
 & 0
\end{array}$$

So,

$$\begin{aligned} P(x) &= x^4-3x^3+6x^2-12x+8 \\ &= (x-2i)(x+2i)(x^2-3x+2) \\ &= (x-2i)(x+2i)(x-2)(x-1) \end{aligned}$$

The zeros of $P(x)$ are $1, 2, 2i,$ and $-2i.$

4. $f(x) = x^4-8x^3+22x^2-28x+16$

The function has degree 4, so there are four zeros. There are three sign changes, so there are either 3 or 1 positive zeros.

$$\begin{aligned} f(-x) &= (-x)^4-8(-x)^3+22(-x)^2-28(-x)+16 \\ &= x^4+8x^3+22x+28x+16 \end{aligned}$$

There are no sign changes in $f(-x)$, so there are no negative zeros. The possible rational zeros are $\{\pm1, \pm2, \pm4, \pm8, \pm16\}$. Using synthetic division to test the positive values, we find that one zero is 2:

$$\begin{array}{r|rrrrr} \underline{2} & 1 & -8 & 22 & -28 & 16 \\ & & 2 & -12 & 20 & -16 \\ \hline & 1 & -6 & 10 & -8 & 0 \end{array}$$

The zeros of the depressed function $x^3-6x^2+10x-8$ are also zeros of *P*. The possible rational zeros of the depressed function are $\{\pm1, \pm2, \pm4, \pm8\}$. Examine only the positive possibilities and find that 4 is a zero:

$$\begin{array}{r|rrrr} \underline{4} & 1 & -6 & 10 & -8 \\ & & 4 & -8 & 8 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

So,

$$\begin{aligned} &x^4+8x^3+22x^2-28x+16 \\ &\qquad = (x-2)(x-4)(x^2-2x+2) \end{aligned}$$

Now find the zeros of x^2-2x+2 using the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2-4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

The zeros are 2, 4, 1 + *i*, and 1 – *i*.

3.5 Basic Concepts and Skills

1. The Fundamental Theorem of Algebra states that a polynomial function of degree $n \geq 1$ has at least one complex zero.

2. The Number of Zeros Theorem states that a polynomial function of degree *n* has exactly $\underline{n}$ zeros, provided a zero of multiplicity *k* is counted $\underline{k}$ times.

3. If *P* is a polynomial function with real coefficients and if $z = a + bi$ is a zero of *P*, then $\underline{\overline{z} = a - bi}$ is also a zero of *P*(*x*).

4. If 2 – 3*i* is a zero of polynomial function with real coefficients, then so is $\underline{2 + 3i}$.

5. False. A polynomial function of degree $n \geq 1$ has at least one complex zero.

6. False. $a, r_1, r_2, \ldots, r_n$ are complex numbers.

7. $x^2+25=0 \Rightarrow x^2=-25 \Rightarrow x=\pm5i$

8. $(x-2)^2+9=0 \Rightarrow (x-2)^2=-9 \Rightarrow$
$x-2=\pm3i \Rightarrow x=2\pm3i$

9. $x^2+4x+4=-9 \Rightarrow (x+2)^2=-9 \Rightarrow$
$x+2=\pm3i \Rightarrow x=-2\pm3i$

10. $x^3-8=0 \Rightarrow (x-2)(x^2+2x+4)=0 \Rightarrow$

$$x=2 \text{ or } x=\frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)}$$
$$=\frac{-2\pm\sqrt{4-16}}{2}=\frac{-2\pm\sqrt{-12}}{2}=-1\pm i\sqrt{3}$$

The solution set is $\left\{2,-1\pm i\sqrt{3}\right\}$.

11. $(x-2)(x-3i)(x+3i)=0 \Rightarrow x=2$ or $x=\pm 3i$

12. $(x-1)(x-2i)(2x-6i)(2x+6i)=0 \Rightarrow$
$x=1$ or $x=2i$ or $x=\pm 3i$

13. The remaining zero is $3-i$.

14. The remaining zero is $2+i$.

15. The remaining zero is $-5-i$.

16. The remaining zeros are i and $1+i$.

17. The remaining zeros are $-i$ and $-3i$.

18. The remaining zeros are $-2i$, $4-i$, and $-1-i$.

19. $$\begin{aligned} P(x)&=2(x-(5-i))(x-(5+i))(x-3i)(x+3i)\\ &=2x^4-20x^3+70x^2-180x+468\end{aligned}$$

20. $$\begin{aligned} P(x)&=-3(x-(2+3i))(x-(2-3i))\\ &\qquad\cdot(x-(1-4i))(x-(1+4i))\\ &=-3x^4+18x^3-114x^2+282x-663\end{aligned}$$

21. $$\begin{aligned} P(x)&=7(x-5)^2(x-1)(x-(3-i))(x-(3+i)\\ &=7x^5-119x^4+777x^3-2415x^2\\ &\qquad+3500x-1750\end{aligned}$$

22. $$\begin{aligned} P(x)&=4(x-3)x^3(x-(2-3i))(x-(2+3i))\\ &=4x^6-28x^5+100x^4-156x^3\end{aligned}$$

23. The function has degree four, so there are four zeros. Since one zero is $3i$, another zero is $-3i$. So $(x-3i)(x+3i)=x^2+9$ is a factor of $P(x)$. Now divide to find the other factor:

$$\begin{array}{r}
x^2+x \\
x^2+9\overline{\smash{)}x^4+x^3+9x^2+9x} \\
\underline{x^4+9x^2} \\
x^3+9x \\
\underline{x^3+9x} \\
0
\end{array}$$

So,

$$\begin{aligned} P(x)&=x^4+x^3+9x^2+9x\\ &=\left(x^2+9\right)\left(x^2+x\right)=x\left(x+1\right)\left(x^2+9\right)\end{aligned}$$

24. The function has degree four, so there are four zeros. Since one zero is $1-i,$ another zero is $1+i.$ So $(x-(1-i))(x-(1+i))=x^2-2x+2$ is a factor of $P(x)$. Now divide to find the other factor:

$$\begin{array}{r}
x^2-1 \\
x^2-2x+2\overline{\smash{)}x^4-2x^3+\ x^2+2x-2} \\
\underline{x^4-2x^3+2x^2} \\
-x^2+2x-2 \\
\underline{-x^2+2x-2} \\
0
\end{array}$$

$$\begin{aligned} P(x)&=x^4-2x^3+\ x^2+2x-2\\ &=\left(x^2-2x+2\right)\left(x^2-1\right)\\ &=\left(x^2-2x+2\right)\left(x-1\right)\left(x+1\right)\end{aligned}$$

25. The function has degree five, so there are five zeros. Since one zero is $3-i,$ another zero is $3+i.$ So $(x-(3-i))(x-(3+i))=x^2-6x+10$ is a factor of $P(x)$. Now divide to find the other factor:

$$\begin{array}{r}
x^3+x^2-2x \\
x^2-6x+10\overline{\smash{)}x^5-5x^4+\ 2x^3+22x^2-20x} \\
\underline{x^5-6x^4+10x^3} \\
x^4-\ 8x^3+22x^2 \\
\underline{x^4-\ 6x^3+10x^2} \\
-2x^3+12x^2-20x \\
\underline{-2x^3+12x^2-20x} \\
0
\end{array}$$

$$\begin{aligned} P(x)&=x^5-5x^4+2x^3+22x^2-20x\\ &=\left(x^2-6x+10\right)\left(x^3+x^2-2x\right)\\ &=\left(x^2-6x+10\right)\left(x^2+x-2\right)x\\ &=x\left(x+2\right)\left(x-1\right)\left(x^2-6x+10\right)\end{aligned}$$

26. The function has degree five, so there are five zeros. Since one zero is *i*, another zero is –*i*. So $(x-i)(x+i)=x^2+1$ is a factor of $P(x)$. Now divide to find the other factor:

$$\begin{array}{r}
2x^3-11x^2+17x-6 \\
x^2+1\overline{\smash{)}2x^5-11x^4+19x^3-17x^2+17x-6} \\
\underline{2x^5 \qquad +2x^3} \qquad\qquad\qquad\qquad \\
-11x^4+17x^3-17x^2 \qquad\qquad \\
\underline{-11x^4 \qquad -11x^2} \qquad\qquad \\
17x^3-6x^2+17x \qquad \\
\underline{17x^3 \qquad +17x} \qquad \\
-6x^2 \qquad -6 \\
\underline{-6x^2 \qquad -6} \\
0
\end{array}$$

$$\begin{aligned}P(x)&=2x^5-11x^4+19x^3-17x^2+17x-6\\&=\left(x^2+1\right)\left(2x^3-11x^2+17x-6\right)\end{aligned}$$

Now find the zeros of $2x^3-11x^2+17x-6.$ There are either three or one positive, rational zeros. The possible zeros are $\left\{\pm\frac{1}{2},\pm1,\pm\frac{3}{2},\pm2,\pm3,\pm6\right\}$. Using synthetic division, we find that $x = 2$ is one of the zeros:

$$\begin{array}{r|rrrr}
\underline{2} & 2 & -11 & 17 & -6 \\
 & & 4 & -14 & 6 \\
\hline
 & 2 & -7 & 3 & 0
\end{array}$$

So,

$$\begin{aligned}P(x)&=2x^5-11x^4+19x^3-17x^2+17x-6\\&=\left(x^2+1\right)\left(2x^3-11x^2+17x-6\right)\\&=\left(x^2+1\right)(x-2)\left(2x^2-7x+3\right)\\&=\left(x^2+1\right)(x-2)(2x-1)(x-3)\end{aligned}$$

27. The function has degree 3, so there are three zeros. There are three sign changes, so there are 1 or 3 positive zeros. The possible rational zeros are $\{\pm1,\pm17\}$. Using synthetic division we find that one zero is 1:

$$\begin{array}{r|rrrr}
\underline{1} & 1 & -9 & 25 & -17 \\
 & & 1 & -8 & 17 \\
\hline
 & 1 & -8 & 17 & 0
\end{array}$$

$x^3-9x^2+25x-17=(x-1)(x^2-8x+17).$

Now solve the depressed equation

$x^2-8x+17=0\Rightarrow$

$$x=\frac{8\pm\sqrt{(-8)^2-4(1)(17)}}{2(1)}\Rightarrow x=\frac{8\pm\sqrt{-4}}{2}\Rightarrow$$

$x=4\pm i.$ The zeros are $1, 4\pm i.$

28. The function has degree 3, so there are three zeros. There are two sign changes, so there are either 2 or 0 positive zeros. There is one sign change in $f(-x)$, so there is one negative zero. The possible rational zeros are $\{\pm1,\pm13\}$. Using synthetic division, we find that one zero is −1:

$$\begin{array}{r|rrrr}
\underline{-1} & 1 & -5 & 7 & 13 \\
 & & -1 & 6 & -13 \\
\hline
 & 1 & -6 & 13 & 0
\end{array}$$

$x^3-5x^2+7x+13=(x+1)(x^2-6x+13).$

Now solve the depressed equation

$x^2-6x+13=0\Rightarrow$

$$x=\frac{6\pm\sqrt{(-6)^2-4(1)(13)}}{2(1)}\Rightarrow x=\frac{6\pm\sqrt{-16}}{2}\Rightarrow$$

$x=3\pm2i.$ The zeros are $-1, 3\pm2i.$

29. The function has degree 3, so there are three zeros. There are two sign changes, so there are either 2 or 0 positive zeros. There is one sign change in $f(-x)$, so there is one negative zero. The possible rational zeros are $\left\{\pm\frac{1}{3},\pm\frac{2}{3},\pm1,\pm\frac{4}{3},\pm\frac{5}{3},\pm2,\pm\frac{8}{3},\pm\frac{10}{3},\pm4,\right.$ $\left.\pm5,\pm\frac{20}{3},\pm8,\pm10,\pm\frac{40}{3},\pm20,\pm40\right\}$. Using synthetic division to test the negative values we find that one zero is $-4/3$.

$$\begin{array}{r|rrrr}
\underline{-\frac{4}{3}} & 3 & -2 & 22 & 40 \\
 & & -4 & 8 & -40 \\
\hline
 & 3 & -6 & 30 & 0
\end{array}$$

$$\begin{aligned}3x^3-2x^2+22x+40&=\left(x+\frac{4}{3}\right)(3x^2-6x+30)\\&=3\left(x+\frac{4}{3}\right)(x^2-2x+10)\end{aligned}$$

Now solve the depressed equation

$x^2-2x+10=0\Rightarrow$

$$x=\frac{2\pm\sqrt{(-2)^2-4(1)(10)}}{2(1)}\Rightarrow x=\frac{2\pm\sqrt{-36}}{2}\Rightarrow$$

$x=1\pm3i.$ The zeros are $-\frac{4}{3}, 1\pm3i.$

30. The function has degree 3, so there are three zeros. There are three sign changes, so there are either 3 or 1 positive zeros. There are no sign changes in $f(-x)$, so there are no negative zeros. The possible rational zeros are $\left\{\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm\frac{4}{3}, \pm 2, \pm 4\right\}$. Using synthetic division to test the positive values we find that one zero is $1/3$:

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -1 & 12 & -4 \\ & & 1 & 0 & 4 \\ \hline & 3 & 0 & 12 & 0 \end{array}$$

$$\begin{aligned} 3x^3 - x^2 + 12x - 4 &= \left(x - \frac{1}{3}\right)(3x^2 + 12) \\ &= 3\left(x - \frac{1}{3}\right)(x^2 + 4) \\ &= 3\left(x - \frac{1}{3}\right)(x - 2i)(x + 2i) \end{aligned}$$

The zeros are $\frac{1}{3}, \pm 2i.$

31. The function has degree 4, so there are four zeros. There are four sign changes, so there are 4, 2, or 0 positive zeros. There are no sign changes in $f(-x)$, so there are no negative zeros. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3, \pm\frac{9}{2}, \pm 9\right\}$. Using synthetic division to test the positive values we find that one zero is 1:

$$\begin{array}{r|rrrrr} 1 & 2 & -10 & 23 & -24 & 9 \\ & & 2 & -8 & 15 & -9 \\ \hline & 2 & -8 & 15 & -9 & 0 \end{array}$$

The zeros of the depressed function $2x^3 - 8x^2 + 15x - 9$ are also zeros of *P*. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3, \pm\frac{9}{2}, \pm 9\right\}$.

Using synthetic division to test the positive values we find that one zero is 1:

$$\begin{array}{r|rrrr} 1 & 2 & -8 & 15 & -9 \\ & & 2 & -6 & 9 \\ \hline & 2 & -6 & 9 & 0 \end{array}$$

Thus,

$$\begin{aligned} &2x^4 - 10x^3 + 23x^2 - 24x + 9 \\ &\quad = (x-1)^2\left(2x^2 - 6x + 9\right). \end{aligned}$$

Now solve the depressed equation $2x^2 - 6x + 9 = 0.$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(2)(9)}}{2(2)} \Rightarrow x = \frac{6 \pm \sqrt{-36}}{4} \Rightarrow$$

$$x = \frac{6 \pm 6i}{4} = \frac{3}{2} \pm \frac{3i}{2}.$$

The zeros of *P* are 1 (multiplicity 2), $\frac{3}{2} \pm \frac{3i}{2}.$

32. The function has degree 4, so there are four zeros. There are two sign changes, so there are either 2 or 0 positive zeros. There are two sign change in $f(-x)$, so there are either 2 or 0 negative zeros. The possible rational zeros are $\left\{\pm\frac{1}{9}, \pm\frac{2}{9}, \pm\frac{1}{3}, \pm\frac{4}{9}, \pm\frac{2}{3}, \pm\frac{8}{9}, \pm 1, \pm\frac{4}{3}, \pm 2, \pm\frac{8}{3}, \pm 4, \pm 8\right\}$. Using synthetic division to test the negative values we find that one zero is –2:

$$\begin{array}{r|rrrrr} -2 & 9 & 30 & 14 & -16 & 8 \\ & & -18 & -24 & 20 & -8 \\ \hline & 9 & 12 & -10 & 4 & 0 \end{array}$$

The zeros of the depressed function $9x^3 + 12x^2 - 10x + 4$ are also zeros of *P*. The possible rational zeros are $\left\{\pm\frac{1}{9}, \pm\frac{2}{9}, \pm\frac{1}{3}, \pm\frac{4}{9}, \pm\frac{2}{3}, \pm 1, \pm\frac{4}{3}, \pm 2, \pm 4\right\}$.

Using synthetic division to test the negative values we find that one zero is –2:

$$\begin{array}{r|rrrr} -2 & 9 & 12 & -10 & 4 \\ & & -18 & 12 & -4 \\ \hline & 9 & -6 & 2 & 0 \end{array}$$

Now solve the depressed equation $9x^2 - 6x + 2 = 0 \Rightarrow$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(9)(2)}}{2(9)} \Rightarrow x = \frac{6 \pm \sqrt{-36}}{18} \Rightarrow$$

$x = \frac{6 \pm 6i}{18} = \frac{1}{3} \pm \frac{1}{3}i.$ The zeros are –2 (multiplicity 2), $\frac{1}{3} \pm \frac{1}{3}i.$

33. The function has degree 4, so there are four zeros. There are three sign changes, so there are either 3 or 1 positive zeros. There is one sign change in $f(-x)$, so there is one negative zero. The possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 5, \pm 6, \pm 10, \pm 15, \pm 30\}$. Using synthetic division to test the negative values, we find that one zero is –3:

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$$\begin{array}{r|rrrrr} \underline{-3} & 1 & -4 & -5 & 38 & -30 \\ & & -3 & 21 & -48 & 30 \\ \hline & 1 & -7 & 16 & -10 & 0 \end{array}$$

The zeros of the depressed function $x^3 - 7x^2 + 16x - 10$ are also zeros of *P*. The possible rational zeros of the depressed function are $\{\pm 1, \pm 2, \pm 5, \pm 10\}$. Since we have already found the negative zero, we examine only the positive possibilities and find that 1 is a zero:

$$\begin{array}{r|rrrr} \underline{1} & 1 & -7 & 16 & -10 \\ & & 1 & -6 & 10 \\ \hline & 1 & -6 & 10 & 0 \end{array}$$

$$x^4 - 4x^3 - 5x^2 + 38x - 30$$
$$= (x+3)(x-1)(x^2 - 6x + 10).$$

Now solve the depressed equation:

$$x^2 - 6x + 10 = 0 \Rightarrow x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)} \Rightarrow$$
$$x = \frac{6 \pm \sqrt{-4}}{2} \Rightarrow x = 3 \pm i.$$

The zeros are –3, 1, $3 \pm i$.

34. The function has degree 4, so there are four zeros. There is one sign change, so there is one positive zero. There are three sign changes in $f(-x)$, so there are either 3 or 1 negative zeros. The possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18\}$. Using synthetic division to test the positive values we find that one zero is 1.

$$\begin{array}{r|rrrrr} \underline{1} & 1 & 1 & 7 & 9 & -18 \\ & & 1 & 2 & 9 & 18 \\ \hline & 1 & 2 & 9 & 18 & 0 \end{array}$$

The zeros of the depressed function $x^3 + 2x^2 + 9x + 18$ are also zeros of *P*. We can factor by grouping to find the next zero:

$$x^3 + 2x^2 + 9x + 18 = x^2(x+2) + 9(x+2)$$
$$= (x+2)(x^2 + 9)$$

So –2 is a zero, as are $\pm 3i$.
The zeros are –2, 1, $\pm 3i$.

35. The function has degree 5, so there are five zeros. There are five sign changes, so there are either 5, 3, or 1 positive zeros. There are no sign changes in $f(-x)$, so there are no negative zeros. The possible rational zeros are $\left\{\frac{1}{2}, 1, \frac{3}{2}, 2, 3, 6\right\}$. Using synthetic division to test the positive values, we find that one zero is $1/2$.

$$\begin{array}{r|rrrrrr} \underline{\frac{1}{2}} & 2 & -11 & 19 & -17 & 17 & -6 \\ & & 1 & -5 & 7 & -5 & 6 \\ \hline & 2 & -10 & 14 & -10 & 12 & 0 \end{array}$$

The zeros of the depressed function $2x^4 - 10x^3 + 14x^2 - 10x + 12$ are also zeros of *P*. Use synthetic division again to find the next zero, 2:

$$\begin{array}{r|rrrrr} \underline{2} & 2 & -10 & 14 & -10 & 12 \\ & & 4 & -12 & 4 & -12 \\ \hline & 2 & -6 & 2 & -6 & 0 \end{array}$$

$$2x^5 - 11x^4 + 19x^3 - 17x^2 + 17x - 6$$
$$= \left(x - \frac{1}{2}\right)(x-2)(2x^3 - 6x^2 + 2x - 6)$$
$$= \left(x - \frac{1}{2}\right)(x-2)(2x^3 - 6x^2 + 2x - 6)$$

Use factoring by grouping to factor $x^3 - 3x^2 + x - 3$:

$$x^3 - 3x^2 + x - 3 = x^2(x-3) + 1(x-3)$$

$= (x^2 + 1)(x - 3)$. So the remaining zeros are 3 and $\pm i$. The zeros are $\frac{1}{2}, 2, 3, \pm i$.

36. The function has degree 5, so there are five zeros. There are four sign changes, so there are either 4, 2, or 0 positive zeros. There is one sign change in $f(-x)$, so there is one negative zero. The possible rational zeros are $\{\pm 1, \pm 2, \pm 4\}$. Using synthetic division to test the negative values, we find that one zero is –2:

$$\begin{array}{r|rrrrrr} \underline{-2} & 1 & -2 & -1 & 8 & -10 & 4 \\ & & -2 & 8 & -14 & 12 & -4 \\ \hline & 1 & -4 & 7 & -6 & 2 & 0 \end{array}$$

The zeros of the depressed function $x^4 - 4x^3 + 7x^2 - 6x + 2$ are also zeros of *P*. Use synthetic division again to find the next zero, 1.

$$\begin{array}{r|rrrrr} \underline{1} & 1 & -4 & 7 & -6 & 2 \\ & & 1 & -3 & 4 & -2 \\ \hline & 1 & -3 & 4 & -2 & 0 \end{array}$$

$$x^5 - 2x^4 - x^3 + 8x^2 - 10x + 4$$
$$= (x+2)(x-1)(x^3 - 3x^2 + 4x - 2).$$

The zeros of the depressed function $x^3 - 3x^2 + 4x - 2$ are also zeros of *P*.

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Use synthetic division again to find the next zero, 1.

$$\begin{array}{r|rrrr} \underline{1} & 1 & -3 & 4 & -2 \\ & & 1 & -2 & 2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$$\begin{aligned} &x^5 - 2x^4 - x^3 + 8x^2 - 10x + 4 \\ &= (x+2)(x-1)(x^3 - 3x^2 + 4x - 2) \\ &= (x+2)(x-1)(x-1)(x^2 - 2x + 2). \end{aligned}$$

Now solve the depressed equation:

$$x^2 - 2x + 2 = 0 \Rightarrow$$
$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

The zeros are $-2, 1$ (multiplicity 2), $1 \pm i$.

37. Since one zero is $3i$, $-3i$ is another zero. There are two zeros, so the degree of the equation is at least 2. Thus, the equation is of the form $f(x) = a(x - 3i)(x + 3i) = a(x^2 + 9)$. The graph passes through (0, 3), so we have $3 = a(0^2 + 9) \Rightarrow a = \frac{1}{3}$. Thus, the equation of the function is $f(x) = \frac{1}{3}(x^2 + 9)$.

38. Since one zero is $-i$, another zero is i. From the graph, we see that 3 is another zero. There are three zeros, so the degree of the equation is at least 3. Thus, the equation is of the form

$$f(x) = a(x-3)(x-i)(x+i) = a(x-3)(x^2+1)$$

The graph passes through (0, −3), so we have $-3 = a(0-3)(0^2+1) \Rightarrow a = 1$. Thus, the equation of the function is

$$f(x) = (x-3)(x^2+1).$$

39. Since i and $2i$ are zeros, so are $-i$, and $-2i$. From the graph, we see that 2 is also a zero. There are five zeros, so the degree of the equation is at least 5. Thus, the equation is of the form

$$\begin{aligned} f(x) &= a(x-i)(x+i)(x-2i)(x+2i)(x-2) \\ &= a(x^2+1)(x^2+4)(x-2) \end{aligned}$$

The graph passes through (0, 4), so we have

$$4 = a(0^2+1)(0^2+4)(0-2) \Rightarrow a = -\frac{1}{2}.$$

Thus, the equation is

$$f(x) = -\frac{1}{2}(x^2+1)(x^2+4)(x-2) \text{ or}$$
$$f(x) = \frac{1}{2}(x^2+1)(x^2+4)(2-x).$$

40. Since $-2i$ is a zero, so is $2i$. From the graph, we see that −1, 0, and 1 are also zeros. There are five zeros, so the degree of the equation is at least 5. Thus, the equation is of the form

$$\begin{aligned} f(x) &= -a(x+2i)(x-2i)(x+1)(x-0)(x-1) \\ &= ax(x^2+4)(1-x^2) \end{aligned}$$

The y-intercept is (0, 0), so we have $0 = a(0)(0^2+4)(0^2-1)$ which is true for all values of a. Thus, the equation is $f(x) = ax(x^2+4)(1-x^2)$, $a \neq 0$.

3.5 Beyond the Basics

41. There are three cube roots. We know that one root is 1. Using synthetic division, we find $0 = x^3 - 1 = (x-1)(x^2+x+1)$:

$$\begin{array}{r|rrrr} \underline{1} & 1 & 0 & 0 & -1 \\ & & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 0 \end{array}$$

Solve the depressed equation

$$x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}.$$

So, the cube roots of 1 are 1 and $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$.

42. The solutions of the equation and the zeros of the polynomial are the same. There are n roots.

43. $P(x) = x^2 + (i-2)x - 2i,\ x = -i$

$$\begin{array}{r|rrr} \underline{-i} & 1 & i-2 & -2i \\ & & -i & 2i \\ \hline & 1 & -2 & 0 \end{array}$$

$$x^2 + (i-2)x - 2i = (x+i)(x-2)$$

44. $P(x) = x^2 + 3ix - 2,\ x = -2i$

$$\begin{array}{r|rrr} \underline{-2i} & 1 & 3i & -2 \\ & & -2i & 2 \\ \hline & 1 & i & 0 \end{array}$$

$$x^2 + 3ix - 2 = (x+2i)(x+i)$$

45. $P(x) = x^3 - (3+i)x^2 - (4-3i)x + 4i,\ x = i$

$$\begin{array}{r|rrrr} i & 1 & -(3+i) & -(4-3i) & 4i \\ & & i & -3i & -4i \\ \hline & 1 & -3 & -4 & 0 \end{array}$$

$$\begin{aligned} &x^3 - (3+i)x^2 - (4-3i)x + 4i \\ &\quad = (x^2 - 3x - 4)(x - i) = (x-4)(x+1)(x-i) \end{aligned}$$

46. $P(x) = x^3 - (4+2i)x^2 + (7+8i)x - 14i,\ x = 2i$

$$\begin{array}{r|rrrr} 2i & 1 & -(4+2i) & 7+8i & -14i \\ & & 2i & -8i & 14i \\ \hline & 1 & -4 & 7 & 0 \end{array}$$

$$\begin{aligned} &x^3 - (4+2i)x^2 + (7+8i)x - 14i \\ &\quad = (x^2 - 4x + 7)(x - 2i) \end{aligned}$$

Now solve $x^2 - 4x + 7 = 0$ to find linear factors.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)} = \frac{4 \pm \sqrt{-12}}{2}$$
$$= \frac{4 \pm 2i\sqrt{3}}{2} = 2 \pm i\sqrt{3}$$

Thus,

$$\begin{aligned} &x^3 - (4+2i)x^2 + (7+8i)x - 14i \\ &\quad = (x^2 - 4x + 7)(x - 2i) \\ &\quad = \left(x - \left(2 + \sqrt{3}\right)\right)\left(x - \left(2 - \sqrt{3}\right)\right)(x - 2i) \\ &\quad = \left(x - 2 - i\sqrt{3}\right)\left(x - 2 + i\sqrt{3}\right)(x - 2i) \end{aligned}$$

47. Since 1 + 2*i* is a zero, so is 1 − 2*i*. The equation is of the form

$$\begin{aligned} f(x) &= a(x-2)(x-(1+2i))(x-(1-2i)) \\ &= a(x-2)(x^2 - 2x + 5) \end{aligned}$$

The *y*-intercept is 40, so we have

$$40 = a(0-2)(0^2 - 2(0) + 5) \Rightarrow a = -4$$

Thus, the equation is

$$f(x) = -4(x-2)(x^2 - 2x + 5).$$

Because $a < 0$, $y \to \infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.

48. Since 2 − 3*i* is a zero, so is 2 + 3*i*. The equation is of the form

$$\begin{aligned} f(x) &= a(x-1)(x-(2-3i))(x-(2+3i)) \\ &= a(x-1)(x^2 - 4x + 13) \end{aligned}$$

The *y*-intercept is −26, so we have

$$-26 = a(0-1)(0^2 - 4(0) + 13) \Rightarrow a = 2$$

Thus, the equation is

$$f(x) = 2(x-1)(x^2 - 4x + 13).$$

Because $a > 0$, $y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$.

49. Since 3 + *i* is a zero, so is 3 − *i*. The equation is of the form

$$\begin{aligned} f(x) &= a(x-1)(x+1)(x-(3+i))(x+(3+i)) \\ &= a(x^2 - 1)(x^2 - 6x + 10) \end{aligned}$$

The *y*-intercept is 20, so we have

$$20 = a(0^2 - 1)(0^2 - 6(0) + 10) \Rightarrow a = -2$$

Thus, the equation is

$$f(x) = -2(x^2 - 1)(x^2 - 6x + 10).$$

Because $a < 0$, $y \to -\infty$ as $x \to -\infty$ and $y \to -\infty$ as $x \to \infty$.

50. Since 1 − 2*i* and 3 − 2*i* are zeros, so are 1 + 2*i* and 3 + 2*i*. The equation is of the form

$$\begin{aligned} f(x) &= a(x-(1-2i))(x-(1+2i)) \cdot \\ &\qquad (x-(3-2i))(x-(3+2i)) \\ &= a(x^2 - 2x + 5)(x^2 - 6x + 13) \end{aligned}$$

The *y*-intercept is 130, so we have

$$130 = a(0^2 - 2(0) + 5)(0^2 - 6(0) + 13) \Rightarrow$$
$$a = 2$$

Thus, the equation is

$$f(x) = 2(x^2 - 2x + 5)(x^2 - 6x + 13).$$

Because $a > 0$, $y \to \infty$ as $x \to -\infty$ and $y \to \infty$ as $x \to \infty$.

3.5 Critical Thinking/Discussion/Writing

51. Factoring the polynomial we have $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ $= a_n(x - r_1)(x - r_2)\cdots(x - r_n)$. Expanding the right side, we find that the coefficient of x^{n-1} is $-a_n r_1 - a_n r_2 - \cdots - a_n r_n$ and the constant term is $(-1)^n a_n r_1 r_2 \cdots r_n$. Comparing the coefficients with those on the left side, we obtain $a_{n-1} = -a_n(r_1 + r_2 + \cdots + r_n)$.

Because $a_n \neq 0$, $-\dfrac{a_{n-1}}{a_n} = r_1 + r_2 + \cdots + r_n$

and $-(1)^n \dfrac{a_0}{a_n} = r_1 r_2 \cdots r_n$.

52. a. $x^3+6x=20 \Rightarrow x^3+6x-20=0.$ There is one sign change, so there is one positive root. There are no sign changes in $f(-x)$, so there are no negative roots. Therefore, there is only one real solution.

b. Substituting $v-u$ for *x* and remembering that $v^3-u^3=20$ and $uv=2,$ we have

$$\begin{aligned}&(v-u)^3+6(v-u)\\&=v^3-3v^2u+3vu^2-u^3+6(v-u)\\&=(v^3-u^3)-(3v^2u-3vu^2)+6(v-u)\\&=(v^3-u^3)-3vu(v-u)+6(v-u)\\&=(v^3-u^3)-(6-3uv)(v-u)\\&=20-(6-3(2))(v-u)=20.\end{aligned}$$

Therefore, $x=v-u$ is the solution.

c. To solve the system $v^3-u^3=20, vu=2$, solve the second equation for *v*, and substitute that value into the first equation, keeping in mind that *u* cannot be zero, so division by *u* is permitted:

$$\begin{aligned}&v^3-u^3=20 \Rightarrow \left(\frac{2}{u}\right)^3-u^3\\&\qquad=\frac{8}{u^3}-u^3=20 \Rightarrow\\&-u^3+\frac{8}{u^3}-20=0.\end{aligned}$$

Let $-u^3=a,$ so

$$a-\frac{8}{a}-20=0 \Rightarrow a^2-20a-8=0 \Rightarrow$$

$$a=\frac{20\pm\sqrt{400+32}}{2}=10\pm6\sqrt{3}=-u^3 \Rightarrow$$

$$\sqrt[3]{-10\pm6\sqrt{3}}=u \Rightarrow v=\frac{2}{\sqrt[3]{-10\pm6\sqrt{3}}}$$

$$=\frac{2}{\sqrt[3]{-10\pm6\sqrt{3}}}\cdot\frac{\sqrt[3]{10\pm6\sqrt{3}}}{\sqrt[3]{10\pm6\sqrt{3}}}=\sqrt[3]{10\pm6\sqrt{3}}$$

d. If *q* is positive, then, according to Descartes's Rule of Signs, the polynomial x^3+px-q has one positive zero and no negative zeros. If *q* is negative, then it has no positive zeros and one negative zero. Either way, there is exactly one real solution. From (b), we have $v^3-u^3=q$, $vu=\frac{p}{3},$ and $x=v-u.$ Substituting, we have

$$\begin{aligned}&x^3+px=(v-u)^3+p(v-u)\\&\quad=v^3-3v^2u+3vu^2-u^3+p(v-u)\\&\quad=(v^3-u^3)-(3v^2u-3vu^2)+p(v-u)\\&\quad=(v^3-u^3)-3vu(v-u)+p(v-u)\\&\quad=(v^3-u^3)+(p-3uv)(v-u)\\&\quad=q+\left(p-3\cdot\frac{p}{3}\right)x=q \Rightarrow x=v-u \text{ is the}\end{aligned}$$

solution.

Solving the system $v^3-u^3=q,\ vu=\frac{p}{3},$ we obtain

$$v^3-u^3=q \Rightarrow \left(\frac{p}{3u}\right)^3-u^3=\frac{p}{3^3u^3}-u^3=q$$

$$\Rightarrow -u^3+\frac{p^3}{3^3u^3}-q=0.$$ Let $-u^3=a,$ so

we have $a-\frac{p}{3^3a}-q=0 \Rightarrow$

$$a^2-qa-\frac{p^3}{3^3}=0 \Rightarrow$$

$$a=\frac{q\pm\sqrt{q^2+4\left(\frac{p^3}{3^3}\right)}}{2}=\frac{q}{2}\pm\sqrt{\frac{q^2+4\left(\frac{p^3}{3^3}\right)}{4}}$$

$$=\frac{q}{2}\pm\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}=-u^3 \Rightarrow$$

$$u=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} \text{ and}$$

$$v=\sqrt[3]{\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} \text{ or}$$

$$u=\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} \text{ and}$$

$$v=\sqrt[3]{\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}.$$

The difference *v* – *u* is the same in both cases, so

$$\begin{aligned}x&=v-u\\&=\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}\\&\qquad-\sqrt[3]{\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}.\end{aligned}$$

e. Substituting $x = y - \frac{a}{3}$, we have

$$x^3 + ax^2 + bx + c = 0 \Rightarrow$$
$$\left(y - \frac{a}{3}\right)^3 + a\left(y - \frac{a}{3}\right)^2 + b\left(y - \frac{a}{3}\right) + c = 0 \Rightarrow$$
$$\left(y^3 - ay^2 + \frac{a^2}{3}y - \frac{a^3}{27}\right) + \left(ay^2 - \frac{2a^2y}{3} + \frac{a^3}{9}\right)$$
$$+ by - \frac{ab}{3} + c = 0 \Rightarrow$$

$y^3 + py = q$, where $p = b - \frac{a^2}{3}$ and
$q = -\frac{2a^3}{27} + \frac{ab}{3} - c$

$$y^3 - \frac{a^2y}{3} + by + \frac{2a^3}{27} - \frac{ab}{3} + c = 0 \Rightarrow$$
$$y^3 + \left(b - \frac{a^2}{3}\right)y = -\frac{2a^3}{27} + \frac{ab}{3} - c$$

53. $x^3 + 6x^2 + 10x + 8 = 0 \Rightarrow a = 6, b = 10, c = 8$.

Substituting $x = y - \frac{6}{3} = y - 2$ as in (1e), we have

$$(y-2)^3 + 6(y-2)^2 + 10(y-2) + 8 = 0 \Rightarrow$$
$$(y^3 - 6y^2 + 12y - 8) + (6y^2 - 24y + 24)$$
$$+10y - 20 + 8 = 0 \Rightarrow$$

$y^3 - 2y + 4 = 0 \Rightarrow y^3 - 2y = -4.$ Then, using the results of (1d), we have

$$y = \sqrt[3]{\frac{-4}{2} + \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-2}{3}\right)^3}}$$
$$- \sqrt[3]{-\frac{-4}{2} + \sqrt{\left(\frac{-4}{2}\right)^2 + \left(\frac{-2}{3}\right)^3}}.$$

Using a calculator, we find that $y = -2$. So $x = -2 - 2 = -4$. Now use synthetic division to find the depressed equation:

$$\begin{array}{r|rrrr} \underline{-4} & 1 & 6 & 10 & 8 \\ & & -4 & -8 & -8 \\ \hline & 1 & 2 & 2 & 0 \end{array}$$

$$x^3 + 6x^2 + 10x + 8 = (x+4)(x^2 + 2x + 2) = 0.$$
$$x^2 + 2x + 2 = 0 \Rightarrow$$
$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm \sqrt{-4}}{2}$$
$$= \frac{-2 \pm 2i}{2} = -1 \pm i.$$

The solution set is $\{-4, -1 \pm i\}$.

3.5 Maintaining Skills

54. $x^2 + 2x = 0 \Rightarrow x(x+2) = 0 \Rightarrow x = 0,\ x = -2$
Solution set: $\{-2, 0\}$

55. $2x^2 + x - 3 = 0 \Rightarrow (2x+3)(x-1) = 0 \Rightarrow$
$x = -\frac{3}{2},\ x = 1$

Solution set: $\left\{-\frac{3}{2}, 1\right\}$

56. $x^3 + 3x^2 - 10x = 0 \Rightarrow x\left(x^2 + 3x - 10\right) = 0 \Rightarrow$
$x(x+5)(x-2) = 0 \Rightarrow x = 0,\ x = -5,\ x = 2$
Solution set: $\{-5, 0, 2\}$

57. $x^4 - x^3 - 12x^2 = 0 \Rightarrow x^2\left(x^2 - x - 12\right) = 0 \Rightarrow$
$x^2(x-4)(x+3) = 0 \Rightarrow x = 0,\ x = 4,\ x = -3$
Solution set: $\{-3, 0, 4\}$

58. $\frac{x^3 - 3x + 1}{x} = x^2 - 3 + \frac{1}{x}$
$Q(x) = x^2 - 3;\ R(x) = 1$

59. $\frac{x^2 - x - 4}{x - 2}$

$$\begin{array}{r|rrr} \underline{2} & 1 & -1 & -4 \\ & & 2 & 2 \\ \hline & 1 & 1 & -2 \end{array}$$

$Q(x) = x + 1;\ R(x) = -2$

60.
$$\begin{array}{r} x \\ x^2 + 0x + 3 \overline{\big)\, x^3 + 0x^2 + 0x + 5} \\ \underline{x^3 + 0x^2 + 3x } \\ -3x + 5 \end{array}$$

$Q(x) = x;\ R(x) = -3x + 5$

61.
$$\begin{array}{r} 4x + 3 \\ 2x^3 + 0x^2 + x \overline{\big)\, 8x^4 + 6x^3 - 0x^2 + 0x - 5} \\ \underline{8x^4 + 0x^3 + 4x^2 } \\ 6x^3 - 4x^2 + 0x \\ \underline{6x^3 + 0x^2 + 3x } \\ -4x^2 - 3x - 5 \end{array}$$

$Q(x) = 4x + 3;\ R(x) = -4x^2 - 3x - 5$

62. $\frac{-3}{2(2) + 1} = -\frac{3}{5}$

63. $\dfrac{7-(-1)}{2(-1)^2+3(-1)}=\dfrac{8}{-1}=-8$

64. $\dfrac{2(-3)+3}{5-2(-3)^2}=\dfrac{-3}{-13}=\dfrac{3}{13}$

65. $\dfrac{(2)^2+4(2)-1}{9-(2)^3}=11$

3.6 Rational Functions

3.6 Practice Problems

1. $f(x)=\dfrac{x-3}{x^2-4x-5}$

The domain of *f* consists of all real numbers for which $x^2-4x-5\neq 0$.

$x^2-4x-5=0 \Rightarrow (x-5)(x+1)=0 \Rightarrow$

$x=5$ or $x=-1$

Thus, the domain is

$(-\infty,-1)\cup(-1,5)\cup(5,\infty)$.

2. a. $g(x)=\dfrac{3}{x-2}$

Let $f(x)=\dfrac{1}{x}$. Then

$g(x)=\dfrac{3}{x-2}=3\left(\dfrac{1}{x-2}\right)=3f(x-2)$.

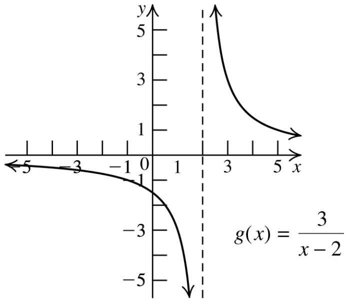
The graph of $y=f(x-2)$ is the graph of $y=f(x)$ shifted two units to the right. This moves the vertical asymptote two units to the right. The graph of $y=3f(x-2)$ is the graph of $y=f(x-2)$ stretched vertically three units.

The domain of *g* is $(-\infty,\ 2)\cup(2,\ \infty)$.

The range of *g* is $(-\infty,\ 0)\cup(0,\ \infty)$.

The vertical asymptote is $x = 2$.

The horizontal asymptote is $y = 0$.



b. $h(x)=\dfrac{2x+5}{x+1}$

$$\begin{array}{r} 2 \\ x+1\overline{)2x+5} \\ \underline{2x+2} \\ 3 \end{array}\qquad h(x)=\dfrac{2x+5}{x+1}=2+\dfrac{3}{x+1}$$

Let $f(x)=\dfrac{1}{x}$. Then

$h(x)=\dfrac{2x+5}{x+1}=2+\dfrac{3}{x+1}=2+3f(x+1)$.

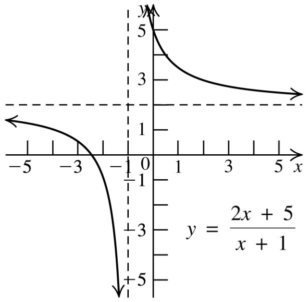
The graph of $y=h(x)$ is the graph of $y=f(x)$ shifted one units to the left and then stretched vertically three units. The graph is then shifted two units up. This moves the vertical asymptote one unit to the left. The horizontal asymptote is shifted two units up.

The domain of *h* is $(-\infty,\ -1)\cup(-1,\ \infty)$.

The range of *h* is $(-\infty,\ 2)\cup(2,\ \infty)$.

The vertical asymptote is $x = -1$.

The horizontal asymptote is $y = 2$.



3. $f(x)=\dfrac{x+1}{x^2+3x-10}$

The vertical asymptotes are located at the zeros of the denominator.

$x^2+3x-10=0 \Rightarrow (x+5)(x-2)=0 \Rightarrow$

$x=-5$ or $x=2$

The vertical asymptotes are $x = -5$ and $x = 2$.

4. $f(x)=\dfrac{3-x}{x^2-9}=\dfrac{-(x-3)}{(x-3)(x+3)}=-\dfrac{1}{x+3}$

$x+3=0 \Rightarrow x=-3$

The vertical asymptote is $x = -3$.

5. a. $f(x)=\dfrac{2x-5}{3x+4}$

Since the numerator and denominator both have degree 1, the horizontal asymptote is $y=\dfrac{2}{3}$.

b. $g(x)=\dfrac{x^2+3}{x-1}$

Since the degree of the numerator is greater than the degree of the denominator, there are no horizontal asymptotes.

c. $h(x)=\dfrac{100x+57}{0.01x^3+8x-9}$

Since the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is the line $y = 0$.

6. $f(x)=\dfrac{2x}{x^2-1}$

There are no common factors of the form $x - a$ between $2x$ and x^2-1.

First find the intercepts:

$$\frac{2x}{x^2-1}=0\Rightarrow x=0$$

The graph passes through the origin.

Find the vertical asymptotes:

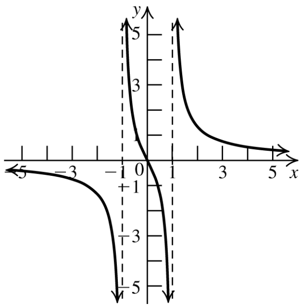
$$x^2-1=0\Rightarrow(x-1)(x+1)=0\Rightarrow$$
$$x=1 \text{ or } x=-1$$

Find the horizontal asymptote: The degree of the numerator is less than the degree of the denominator, so the horizontal asymptote is the x-axis.

By long division, $f(x)=\dfrac{2x}{x^2-1}=0+\dfrac{2x}{x^2-1}$.

$R(x)=2x$ has zero 0 and $D(x)=x^2-1$ has zeros −1 and 1. These zeros divide the x-axis into four intervals, $(-\infty,-1)$, $(-1,0)$, $(0,1)$, and $(1,\infty)$. Use test values to determine where the graph of f is above and below the x-axis.

Interval	Test point	Value of $f(x)$	Above/below x-axis
$(-\infty,-1)$	−3	$-\frac{3}{4}$	below
$(-1,0)$	$-\frac{1}{2}$	$\frac{4}{3}$	above
$(0,1)$	$\frac{1}{2}$	$-\frac{4}{3}$	below
$(1,\infty)$	2	$\frac{4}{3}$	above



7. $f(x)=\dfrac{2x^2-1}{2x^2+x-3}$

$f(x)$ is in lowest terms.

Find the intercepts:

$$\frac{2x^2-1}{2x^2+x-3}=0\Rightarrow x=\pm\frac{\sqrt{2}}{2};\ f(0)=\frac{1}{3}$$

Find the vertical asymptotes:

$$2x^2+x-3=0\Rightarrow(2x+3)(x-1)=0\Rightarrow$$
$$x=-\frac{3}{2} \text{ or } x=1$$

Find the horizontal asymptote: The degree of the numerator is the same as the degree of the denominator, so the horizontal asymptote is $y=\dfrac{2}{2}=1$.

$$f(x)=\frac{2x^2-1}{2x^2+x-3}=1+\frac{2-x}{2x^2+x-3}$$

The zero of $R(x)=2-x$ is 2 and the zeros of $D(x)=2x^2+x-3=(2x+3)(x-1)$ are $x=-\dfrac{3}{2}$ and $x = 1$.

Use test values to determine where the graph of f is above and below the horizontal asymptote y = 1.

Interval	Test point	Value of $f(x)$	Above/below $y = 1$
$\left(-\infty,-\frac{3}{2}\right)$	−2	$\frac{7}{3}$	above
$\left(-\frac{3}{2},1\right)$	0	$\frac{1}{3}$	below
$(1,2)$	$\frac{3}{2}$	$\frac{7}{6}$	above
$(2,\infty)$	3	$\frac{17}{18}$	above

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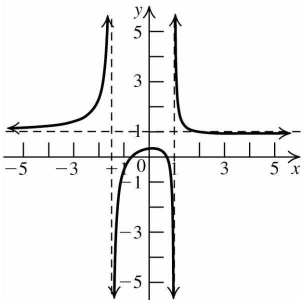


Notice that the graph crosses the horizontal asymptote at (2, 1).

8. $f(x)=\dfrac{x^2+1}{x^2+2}$

$f(x)$ is in lowest terms.

Find the intercepts:

$\dfrac{x^2+1}{x^2+2}=0 \Rightarrow x=\pm i \Rightarrow$ there is no x-intercept.

$f(0)=\dfrac{1}{2} \Rightarrow \left(0, \dfrac{1}{2}\right)$ is the y-intercept.

Find the vertical asymptotes:

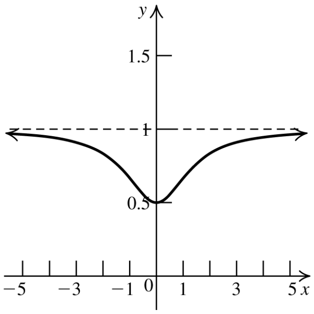
$x^2+2=0 \Rightarrow x=i\sqrt{2} \Rightarrow$ there is no vertical asymptote.

Find the horizontal asymptote: The degree of the numerator is the same as the degree of the denominator, so the horizontal asymptote is

$y=\dfrac{1}{1}=1$.

$f(x)=\dfrac{x^2+1}{x^2+2}=1+\dfrac{-1}{x^2+2}$

Neither $R(x)=-1$ nor $D(x)=x^2+2$ have real zeros. Since $\dfrac{-1}{x^2+2}$ is negative for all values of x, the graph of $f(x)$ is always below the line $y = 1$.



9. $f(x)=\dfrac{x^2+2}{x-1}$

$f(x)$ is in lowest terms.

Find the intercepts:

$\dfrac{x^2+2}{x-1}=0 \Rightarrow x=\pm i\sqrt{2} \Rightarrow$ there is no x-intercept. $f(0)=-2 \Rightarrow (0,-2)$ is the y-intercept.

Find the vertical asymptotes:

$x-1=0 \Rightarrow x=1$

Find the horizontal asymptote: The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote. However, there is an oblique asymptote.

$\dfrac{x^2+2}{x-1}=x+1+\dfrac{3}{x-1} \Rightarrow y=x+1$ is the oblique asymptote.

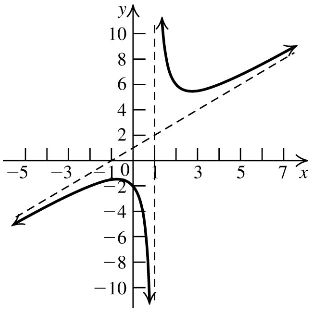
The graph is above the line $y = x + 1$ on $(1, \infty)$ and below the line on $(-\infty, 1)$.

The intervals determined by the zeros of the numerator and of the denominator of

$f(x)-(x+1)=\dfrac{x^2+2}{x-1}-(x+1)=\dfrac{3}{x-1}$ divide the x-axis into two intervals, $(-\infty,1)$ and $(1,\infty)$.

Use test numbers to determine where the graph of f is above and below the x-axis.

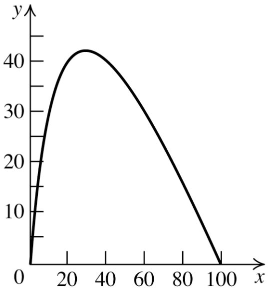
Interval	Test point	Value of $f(x)$	Above/below $y = x + 1$
$(-\infty,1)$	-1	$-\frac{3}{2}$	below
$(1,\infty)$	2	6	above



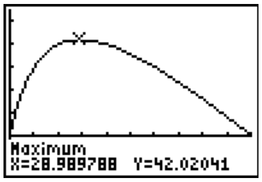
10. $R(x) = \dfrac{x(100-x)}{x+20}$

a. $R(10) = \dfrac{10(100-10)}{10+20} = 30$ billion dollars.
This means that if income is taxed at a rate of 10%, then the total revenue for the government will be 30 billion dollars. Similarly, $R(20)$ = \$40 billion, $R(30)$ = \$42 billion, $R(40)$ = \$40 billion, $R(50) \approx$ \$35.7 billion, $R(60)$ = \$30 billion.

b.



c.



```
Maximum
X=28.989788  Y=42.02041
```

From the graphing calculator screen, we see that a tax rate of about 29% generates the maximum tax revenue of about \$42.02 billion.

3.6 Basic Concepts and Skills

1. A rational function can be expressed in the form $\dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

2. The line $x = a$ is a vertical asymptote of f if $|f(x)| \to \infty$ as $x \to \underline{a^+}$ or as $x \to \underline{a^-}$.

3. The line $y = k$ is a horizontal asymptote of f if $f(x) \to k$ as $x \to \underline{\infty}$ or as $x \to \underline{-\infty}$.

4. If an asymptote is neither horizontal nor vertical, then it is called an oblique asymptote.

5. False. A rational function has a vertical asymptote only if the denominator has a real zero and the numerator and denominator have no common factors.

6. True

7. $(-\infty,-4)\cup(-4,\infty)$ **8.** $(-\infty,1)\cup(1,\infty)$

9. $(-\infty,\infty)$ **10.** $(-\infty,\infty)$

11. $x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = -2, 3$
The domain of the function is $(-\infty,-2)\cup(-2,3)\cup(3,\infty)$.

12. $x^2 - 6x - 7 = 0 \Rightarrow (x-7)(x+1) = 0 \Rightarrow x = -1, 7$
The domain of the function is $(-\infty,-1)\cup(-1,7)\cup(7,\infty)$.

13. $x^2 - 6x + 8 = 0 \Rightarrow (x-4)(x-2) = 0 \Rightarrow x = 2, 4$
The domain of the function is $(-\infty,2)\cup(2,4)\cup(4,\infty)$.

14. $x^2 - 3x + 2 = 0 \Rightarrow (x-2)(x-1) = 0 \Rightarrow x = 1, 2$
The domain of the function is $(-\infty,1)\cup(1,2)\cup(2,\infty)$.

15. As $x \to 1^+, f(x) \to \infty$.

16. As $x \to 1^-, f(x) \to \infty$.

17. As $x \to -2^+, f(x) \to \infty$.

18. As $x \to -2^-, f(x) \to -\infty$.

19. As $x \to \infty, f(x) \to 1$.

20. As $x \to -\infty, f(x) \to 1$.

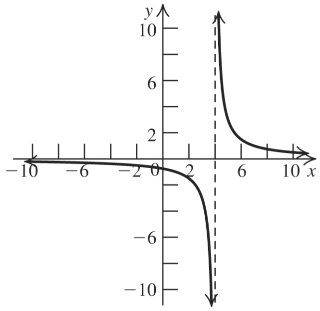
21. The domain of f is $(-\infty,-2)\cup(-2,1)\cup(1,\infty)$.

22. There are 2 vertical asymptotes.

23. The equations of the vertical asymptotes of the graph are $x = -2$ and $x = 1$.

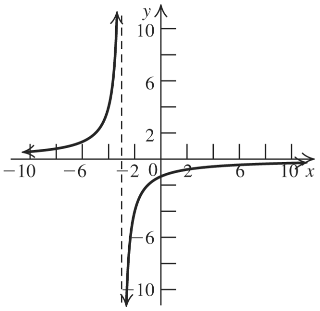
24. The equation of the horizontal asymptote of the graph is $y = 1$.

25.



Domain: $(-\infty,4)\cup(4,\infty)$
Range: $(-\infty,\ 0)\cup(0,\ \infty)$
Vertical asymptote: $x = 4$
Horizontal asymptote: $y = 0$

26.

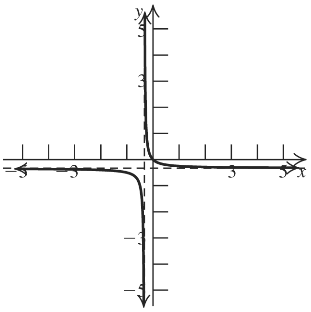


Domain: $(-\infty,-3)\cup(-3,\infty)$

Range: $(-\infty,\ 0)\cup(0,\ \infty)$

Vertical asymptote: $x=-3$
Horizontal asymptote: $y=0$

27.



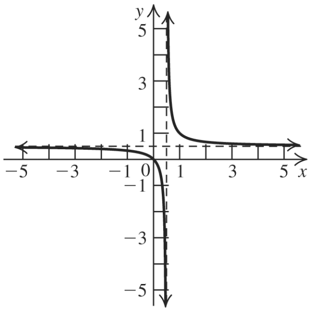
Domain: $\left(-\infty,-\frac{1}{3}\right)\cup\left(-\frac{1}{3},\infty\right)$

Range: $\left(-\infty,-\frac{1}{3}\right)\cup\left(-\frac{1}{3},\infty\right)$

Vertical asymptote: $x=-\frac{1}{3}$

Horizontal asymptote: $y=-\frac{1}{3}$

28.



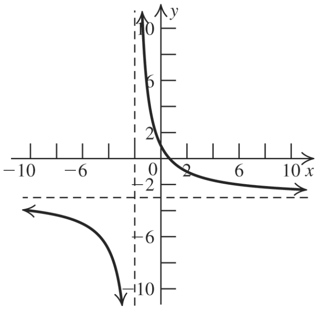
Domain: $\left(-\infty,\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$

Range: $\left(-\infty,\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$

Vertical asymptote: $x=\frac{1}{2}$

Horizontal asymptote: $y=\frac{1}{2}$

29.

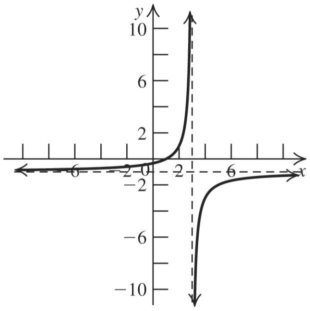


Domain: $(-\infty,-2)\cup(-2,\infty)$

Range: $(-\infty,\ -3)\cup(-3,\ \infty)$

Vertical asymptote: $x=-2$
Horizontal asymptote: $y=-3$

30.

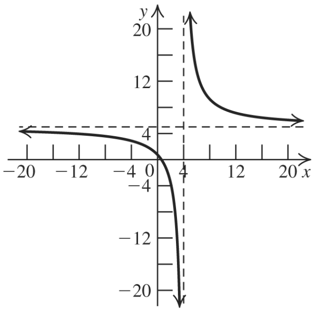


Domain: $(-\infty,\ 3)\cup(3,\ \infty)$

Range: $(-\infty,\ -1)\cup(-1,\ \infty)$

Vertical asymptote: $x=3$
Horizontal asymptote: $y=-1$

31.

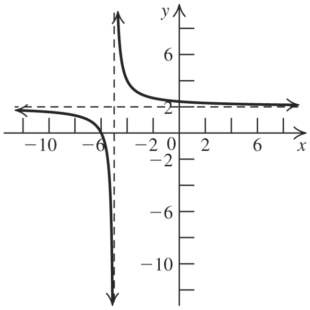


Domain: $(-\infty,\ 4)\cup(4,\infty)$

Range: $(-\infty,\ 5)\cup(5,\ \infty)$

Vertical asymptote: $x=4$
Horizontal asymptote: $y=5$

32.



Domain: $(-\infty, -5) \cup (-5, \infty)$
Range: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = -5$
Horizontal asymptote: $y = 2$

In exercises 33–42, to find the vertical asymptotes, first eliminate any common factors in the numerator and denominator, and then set the denominator equal to zero and solve for x.

33. $x = 1$ **34.** $x = 2$

35. $x = -4, x = 3$ **36.** $x = -\frac{3}{2}, x = \frac{4}{3}$

37. $h(x) = \frac{x^2 - 1}{x^2 + x - 6} = \frac{(x-1)(x+1)}{(x-2)(x+3)}.$
The equations of the vertical asymptotes are $x = -3$ and $x = 2$.

38. $h(x) = \frac{x^2 - 4}{3x^2 + x - 4} = \frac{(x-2)(x+2)}{(3x+4)(x-1)}.$
The equations of the vertical asymptotes are $x = -\frac{4}{3}$ and $x = 1$.

39. $f(x) = \frac{x^2 - 6x + 8}{x^2 - x - 12} = \frac{(x-4)(x-2)}{(x-4)(x+3)}.$
Disregard the common factor. The vertical asymptote is $x = -3$.

40. $f(x) = \frac{x^2 - 9}{x^3 - 4x} = \frac{(x-3)(x+3)}{x(x-2)(x+2)}.$
The equations of the vertical asymptotes are $x = 0, x = 2,$ and $x = -2$.

41. There is no vertical asymptote.

42. There is no vertical asymptote.

For exercises 43–50, locate the horizontal asymptote as follows:

- If the degree of the numerator of a rational function is less than the degree of the denominator, then the x-axis ($y = 0$) if the horizontal asymptote.
- If the degree of the numerator of a rational function equals the degree of the denominator, the horizontal asymptote is the line with the equation $y = \frac{a_n}{b_m}$, where a_n is the coefficient of the leading term of the numerator and b_m is the coefficient of the leading term of the denominator.
- If the degree of the numerator of a rational function is greater than the degree of the denominator, then there is no horizontal asymptote.

43. $y = 0$ **44.** $y = 0$

45. $y = \frac{2}{3}$ **46.** $y = -\frac{3}{4}$

47. There is no horizontal asymptote.

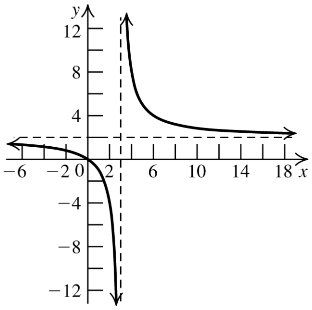
48. $y = 0$ **49.** $y = 0$

50. There is no horizontal asymptote.

51. d **52.** f **53.** e

54. b **55.** a **56.** c

57. $0 = \frac{2x}{x-3} \Rightarrow x = 0$ is the x-intercept.
$\frac{2(0)}{0-3} = 0 \Rightarrow y = 0$ is the y-intercept. The vertical asymptote is $x = 3$. The horizontal asymptote is $y = 2$. The intervals to be tested are $(-\infty, 3)$ and $(3, \infty)$. The graph is above the horizontal asymptote on $(3, \infty)$ and below the horizontal asymptote on $(-\infty, 3)$.



58. $0 = \frac{-x}{x-1} \Rightarrow x = 0$ is the x-intercept.
$\frac{0}{0-1} = 0 \Rightarrow y = 0$ is the y-intercept. The vertical asymptote is $x = 1$. The horizontal asymptote is $y = -1$. The intervals to be tested are $(-\infty, 1),$ and $(1, \infty)$.

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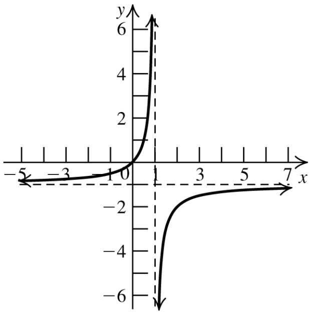
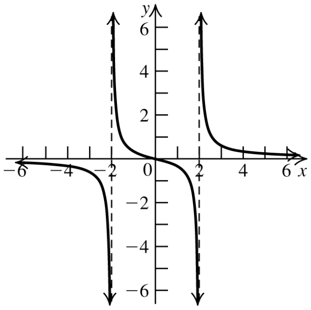
The graph is above the horizontal asymptote on $(-\infty, 1)$ and below the horizontal asymptote on $(1, \infty)$.



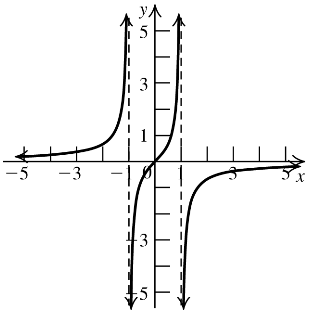
59. $0 = \dfrac{x}{x^2 - 4} \Rightarrow x = 0$ is the x-intercept.

$\dfrac{0}{0^2 - 4} = 0 \Rightarrow y = 0$ is the y-intercept.
The vertical asymptotes are $x = -2$ and $x = 2$.
The horizontal asymptote is the x-axis.
The intervals to be tested are $(-\infty, -2), (-2, 0), (0, 2)$ and $(2, \infty)$. The graph is above the x-axis on $(-2, 0) \cup (2, \infty)$ and below the x-axis on $(-\infty, -2) \cup (0, 2)$.



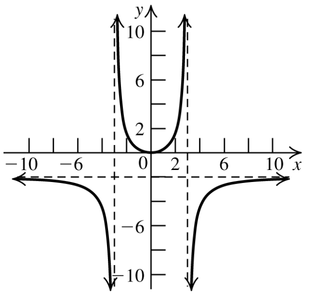
60. $0 = \dfrac{x}{1 - x^2} \Rightarrow x = 0$ is the x-intercept.

$\dfrac{0}{1 - 0^2} = 0 \Rightarrow y = 0$ is the y-intercept. The vertical asymptotes are $x = -1$ and $x = 1$. The horizontal asymptote is the x-axis.
The intervals to be tested are $(-\infty, -1), (-1, 0), (0, 1)$ and $(1, \infty)$. The graph is above the x-axis on $(-\infty, -1) \cup (0, 1)$ and below the x-axis on $(-1, 0) \cup (1, \infty)$.



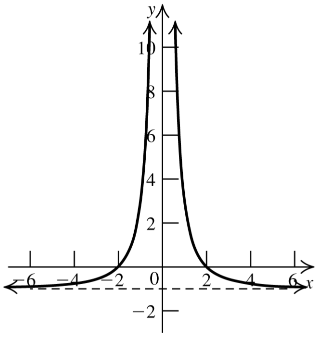
61. $0 = \dfrac{-2x^2}{x^2 - 9} \Rightarrow x = 0$ is the x-intercept.

$\dfrac{0^2}{0^2 - 4} = 0 \Rightarrow y = 0$ is the y-intercept. The vertical asymptotes are $x = -3$ and $x = 3$.
The horizontal asymptote is $y = -2$.
The intervals to be tested are $(-\infty, -3), (-3, 3)$, and $(3, \infty)$. The graph is above the horizontal asymptote on $(-3, 3)$ and below the horizontal asymptote on $(-\infty, -3) \cup (3, \infty)$.



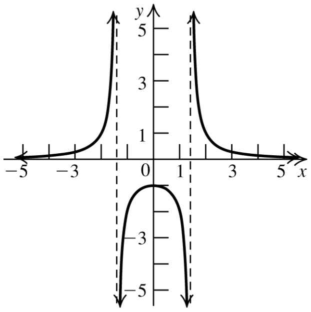
62. $0 = \dfrac{4 - x^2}{x^2} \Rightarrow x = \pm 2$ is the x-intercept.

$\dfrac{4 - 0^2}{0^2} \Rightarrow$ there is no y-intercept. The vertical asymptote is the y-axis. The horizontal asymptote is $y = -1$. The intervals to be tested are $(-\infty, 0)$, and $(0, \infty)$. The graph is above the horizontal asymptote on $(-\infty, 0) \cup (0, \infty)$.



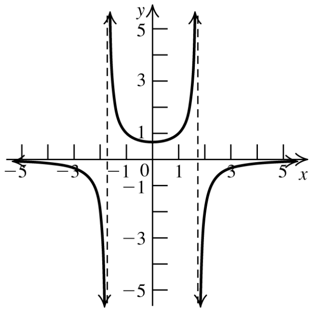
63. $0 = \dfrac{2}{x^2 - 2} \Rightarrow$ there is no x-intercept.

$\dfrac{2}{0^2 - 2} = 0 \Rightarrow y = -1$ is the y-intercept. The vertical asymptotes are $x = \pm\sqrt{2}$. The horizontal asymptote is the x-axis. The intervals to be tested are $(-\infty, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, \infty)$. The graph is above the x-axis on $\left(-\infty, -\sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$ and below the x-axis on $\left(-\sqrt{2}, \sqrt{2}\right)$.



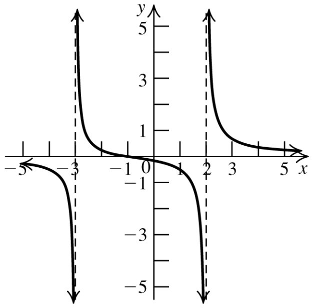
64. $0 = \dfrac{-2}{x^2 - 3} \Rightarrow$ there is no x-intercept.

$\dfrac{-2}{0^2 - 3} = 0 \Rightarrow y = \dfrac{2}{3}$ is the y-intercept. The vertical asymptotes are $x = \pm\sqrt{3}$. The horizontal asymptote is the x-axis. The intervals to be tested are $(-\infty, -\sqrt{3}), (-\sqrt{3}, \sqrt{3})$ and $(\sqrt{3}, \infty)$. The graph is above the x-axis on $\left(-\sqrt{3}, \sqrt{3}\right)$ and below the x-axis on $\left(-\infty, -\sqrt{3}\right) \cup \left(\sqrt{3}, \infty\right)$.



65. $0 = \dfrac{x+1}{(x-2)(x+3)} \Rightarrow x = -1$ is the x-intercept.

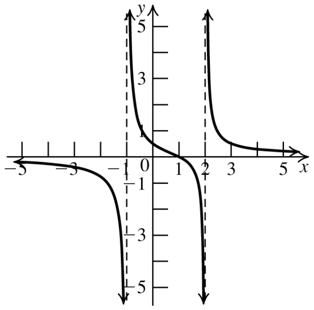
$\dfrac{0+1}{(0-2)(0+3)} = -\dfrac{1}{6} \Rightarrow y = -\dfrac{1}{6}$ is the y-intercept. The vertical asymptotes are $x = -3$ and $x = 2$. The horizontal asymptote is the x-axis. The intervals to be tested are $(-\infty, -3), (-3, -1),\ (-1, 2),$ and $(2, \infty)$. The graph is above the x-axis on $(-3, -1) \cup (2, \infty)$ and below the x-axis on $(-\infty, -3) \cup (-1, 2)$.



66. $0 = \dfrac{x-1}{(x+1)(x-2)} \Rightarrow x = 1$ is the x-intercept.

$\dfrac{0-1}{(0+1)(0-2)} = \dfrac{1}{2} \Rightarrow y = \dfrac{1}{2}$ is the y-intercept. The vertical asymptotes are $x = -1$ and $x = 2$. The horizontal asymptote is the x-axis. The intervals to be tested are $(-\infty, -1), (-1, 1),$ $(1, 2),$ and $(2, \infty)$.

The graph is above the x-axis on $(-1, 1) \cup (2, \infty)$ and below the x-axis on $(-\infty, -1) \cup (1, 2)$.



67. $0 = \dfrac{x^2}{x^2 + 1} \Rightarrow x = 0$ is the x-intercept.

$\dfrac{0^2}{0^2 + 1} = 0 \Rightarrow y = 0$ is the y-intercept. There is no vertical asymptote. The horizontal asymptote is $y = 1$. The intervals to be tested are $(-\infty, 0)$ and $(0, \infty)$. The graph is above the x-axis on $(-\infty, 0) \cup (0, \infty)$ and below the horizontal asymptote on $\left(-\infty,\ \infty\right)$.

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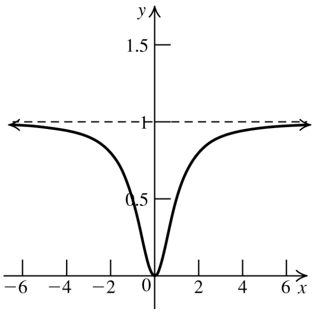
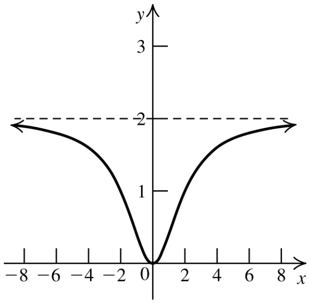


68. $0=\dfrac{2x^2}{x^2+4}\Rightarrow x=0$ is the x-intercept.

$\dfrac{2(0^2)}{0^2+4}=0\Rightarrow y=0$ is the y-intercept. There is no vertical asymptote. The horizontal asymptote is $y = 2$. The intervals to be tested are $(-\infty,0)$ and $(0,\infty)$. The graph is above the x-axis on $(-\infty,0)\cup(0,\infty)$ and below the horizontal asymptote on $(-\infty,\infty)$.



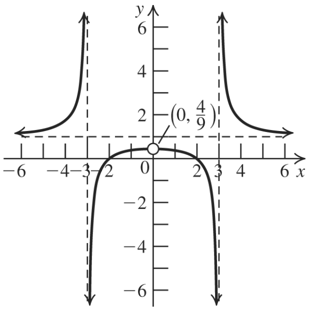
69. $f(x)=\dfrac{x^3-4x}{x^3-9x}=\dfrac{x(x-2)(x+2)}{x(x-3)(x+3)}$
$=\dfrac{(x-2)(x+2)}{(x-3)(x+3)}$

$f(x)=0\Rightarrow x=\pm2$ are the x-intercepts.

$\dfrac{x^3-4x}{x^3-9x}=\dfrac{x\left(x^2-4\right)}{x\left(x^2-9\right)}=\dfrac{x^2-4}{x^2-9}\Rightarrow$

$\dfrac{0^2-4}{0^2-9}=\dfrac{4}{9}\Rightarrow\dfrac{4}{9}$ is the y-intercept. However, there is a hole at $\left(0,\ \dfrac{4}{9}\right)$ since $0^3-9(0)=0$

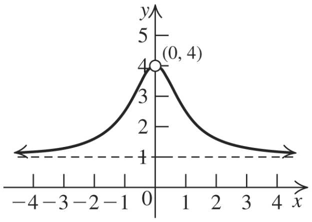
$x^2-9=0\Rightarrow x(x+3)(x-3)=0\Rightarrow x=-3$ and $x = 3$ are the vertical asymptotes. The degree of the numerator is the same as the degree of the denominator, so the horizontal asymptote is $y = 1$. The intervals to be tested are $(-\infty,-3),(-3,\ 0),\ (0,\ 3),$ and $(3,\infty)$. The graph is above the horizontal asymptote on $(-\infty,-3)\cup(3,\infty)$ and below the horizontal asymptote on $(-3,\ 0)\cup(0,\ 3)$.



70. $f(x)=\dfrac{x^3+32x}{x^3+8x}=\dfrac{x\left(x^2+32\right)}{x\left(x^2+8\right)}=\dfrac{x^2+32}{x^2+8}$

$0=\dfrac{x^2+32}{x^2+8}\Rightarrow x^2=-32\Rightarrow$ there are no x-intercepts.

$\dfrac{0^2+32}{0^2+8}=\dfrac{32}{8}\Rightarrow y=4$ is the y-intercept. However, there is a hole at (0, 4) since $0^3+8(0)=0.$ Since there is no real solution for $x^2+8=0,$ there are no vertical asymptotes. The horizontal asymptote is $y = 1$ since the degrees of the numerator and the denominator are equal and the leading coefficients are the same. The intervals to be tested are $(-\infty,0)$ and $(0,\infty)$. The graph is above the horizontal asymptote on $(-\infty,0)\cup(0,\infty)$.



71. $0=\dfrac{(x-2)^2}{x-2}\Rightarrow$ there is no x-intercept. There is a hole at (2, 0). $\dfrac{(0-2)^2}{0-2}=-2\Rightarrow y=-2$ is the y-intercept. There are no vertical asymptotes. There are no horizontal asymptotes. The intervals to be tested are $(-\infty,2)$ and $(2,\infty)$. The graph is above the x-axis on $(2,\infty)$ and below the x-axis on $(-\infty,2)$.

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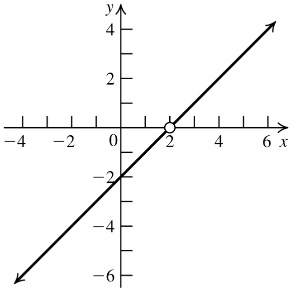
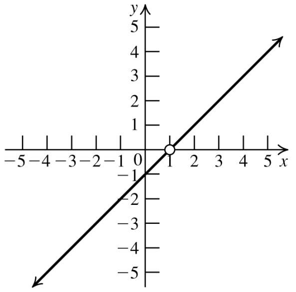


72. $0 = \frac{(x-1)^2}{x-1} \Rightarrow$ there is no x-intercept. There is a hole at (1, 0). $\frac{(0-1)^2}{0-1} = -1 \Rightarrow y = -1$ is the y-intercept. There are no vertical asymptotes. There are no horizontal asymptotes. The intervals to be tested are $(-\infty, 1)$ and $(1, \infty)$. The graph is above the x-axis on $(1, \infty)$ and below the x-axis on $(-\infty, 1)$.



73. The x-intercept is 1 and the vertical asymptote is $x = 2$. The horizontal asymptote is $y = -2$, so the degree of the numerator equals the degree of the denominator, and the ratio of the leading terms of the numerator and the denominator is −2. Thus, the equation is of the form $y = a\left(\frac{x-1}{x-2}\right)$. The y-intercept is −1, so we have $-1 = a\left(\frac{0-1}{0-2}\right) \Rightarrow a = -2$.

Thus, the equation is $f(x) = \frac{-2(x-1)}{x-2}$.

74. The x-intercept is −2 and the vertical asymptote is $x = 1$. The horizontal asymptote is $y = \frac{1}{2}$, so the degree of the numerator equals the degree of the denominator, and the ratio of the leading terms of the numerator and the denominator is $\frac{1}{2}$.

Thus, the equation is of the form $y = a\left(\frac{x+2}{x-1}\right)$.

The y-intercept is −1, so we have $-1 = a\left(\frac{0+2}{0-1}\right) \Rightarrow a = \frac{1}{2}$.

Thus, the equation is $f(x) = \frac{x+2}{2(x-1)}$.

75. The x-intercepts are 1 and 3, and the vertical asymptotes are $x = 0$ and $x = 2$. The horizontal asymptote is $y = 1$, so the degree of the numerator equals the degree of the denominator, and the ratio of the leading terms of the numerator and the denominator is 1. Thus, the equation is of the form $y = a\frac{(x-1)(x-3)}{x(x-2)}$.

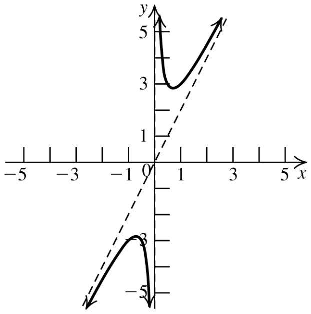
There is no y-intercept, so the equation is $f(x) = \frac{(x-1)(x-3)}{x(x-2)}$.

76. The x-intercepts are −2 and 3. The vertical asymptotes are $x = -1$ and $x = 2$. The horizontal asymptote is $y = 1$, so the degree of the numerator equals the degree of the denominator, and the ratio of the leading terms of the numerator and the denominator is 1. Thus, the equation is of the form $y = a\frac{(x+2)(x-3)}{(x+1)(x-2)}$.

The y-intercept is 3, so we have $3 = a \cdot \frac{(0+2)(0-3)}{(0+1)(0-2)} \Rightarrow a = 1$. Thus, the equation is $f(x) = \frac{(x+2)(x-3)}{(x+1)(x-2)}$.

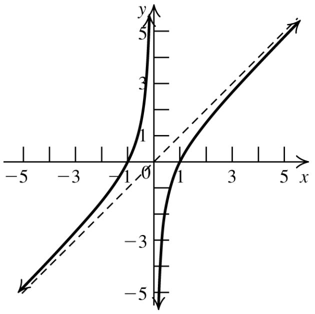
77. $\frac{2x^2+1}{x} = 2x + \frac{1}{x}$.

The oblique asymptote is $y = 2x$.



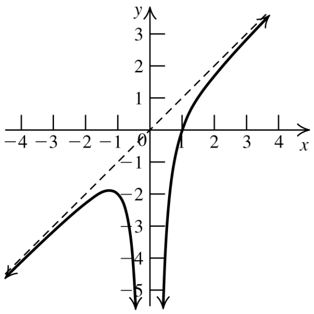
78. $\dfrac{x^2-1}{x}=x-\dfrac{1}{x}.$

The oblique asymptote is $y=x$.



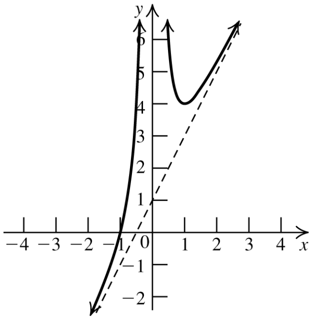
79. $\dfrac{x^3-1}{x^2}=x-\dfrac{1}{x^2}.$

The oblique asymptote is $y=x$.



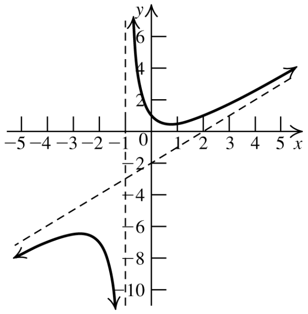
80. $\dfrac{2x^3+x^2+1}{x^2}=2x+1+\dfrac{1}{x^2}.$

The oblique asymptote is $y=2x+1$.



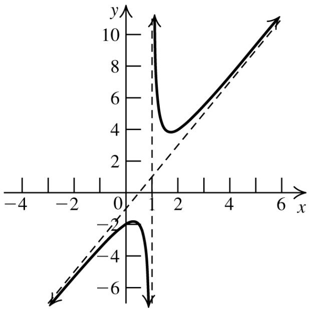
81. $$\begin{array}{r} x-2 \\ x+1\overline{)x^2-x+1} \\ \underline{x^2+x} \\ -2x+1 \\ \underline{-2x-2} \\ 3 \end{array}$$

The oblique asymptote is $y=x-2$.



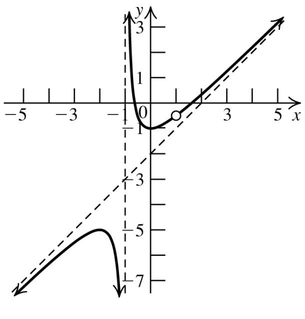
82. $$\begin{array}{r} 2x-1 \\ x-1\overline{)2x^2-3x+2} \\ \underline{2x^2-2x} \\ -x+2 \\ \underline{-x+1} \\ 1 \end{array}$$

The oblique asymptote is $y=2x-1$.



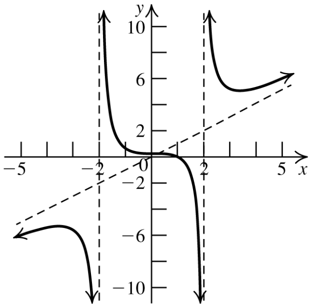
83. $$\begin{array}{r} x-2 \\ x^2-1\overline{)x^3-2x^2+0x+1} \\ \underline{x^3\qquad -x} \\ -2x^2+x+1 \\ \underline{-2x^2\qquad +2} \\ x-1 \end{array}$$

The oblique asymptote is $y=x-2$. Note that there is a hole in the graph at $x=1$.



84.
$$\begin{array}{r} x \\ x^2-4\overline{\smash{)}x^3+0x^2+0x-1} \\ \underline{x^3 \qquad\quad -4x} \\ 4x-1 \end{array}$$

The oblique asymptote is $y = x$.



3.6 Applying the Concepts

85. a. $C(x) = 0.5x + 2000$

b. $\overline{C}(x) = \dfrac{C(x)}{x} = \dfrac{0.5x + 2000}{x} = 0.5 + \dfrac{2000}{x}$

c. $\overline{C}(100) = 0.5 + \dfrac{2000}{100} = 20.5$

$\overline{C}(500) = 0.5 + \dfrac{2000}{500} = 4.5$

$\overline{C}(1000) = 0.5 + \dfrac{2000}{1000} = 2.5$

These show the average cost of producing 100, 500, and 1000 trinkets, respectively.

d. The horizontal asymptote of $\overline{C}(x)$ is $y = 0.5$. It means that the average cost approaches the daily fixed cost of producing each trinket as the number of trinkets approaches ∞.

86. a. $\overline{C}(x) = \dfrac{-0.002x^2 + 6x + 7000}{x}$

$= -0.002x + 6 + \dfrac{7000}{x}$

b. $\overline{C}(100) = -0.002(100) + 6 + \dfrac{7000}{100} = 75.8$

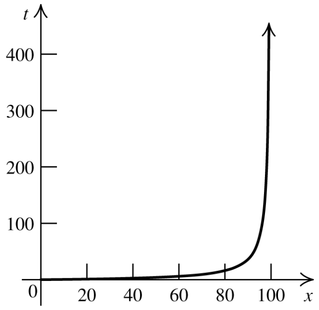
$\overline{C}(500) = -0.002(500) + 6 + \dfrac{7000}{500} = 19$

$\overline{C}(1000) = -0.002(1000) + 6 + \dfrac{7000}{1000} = 11$

These show the average cost of producing 100, 500, and 1000 CD players, respectively.

c. The oblique asymptote is $y = -0.002x + 6.$ For large values of *x*, this is a good approximation of the average cost of producing *x* CD players.

87. a.



b. $f(50) = \dfrac{4(50)+1}{100-50} \approx 4$ min

$f(75) = \dfrac{4(75)+1}{100-75} \approx 12$ min

$f(95) = \dfrac{4(95)+1}{100-95} \approx 76$ min

$f(99) = \dfrac{4(99)+1}{100-99} \approx 397$ min

c. **(i)** As $x \to 100^-, f(x) \to \infty.$

(ii) The statement is not applicable because the domain is $x < 100$.

d. No, the bird doesn't ever collect all the seed from the field.

88. a. $C(50) = \dfrac{1000}{100-50} = 20$

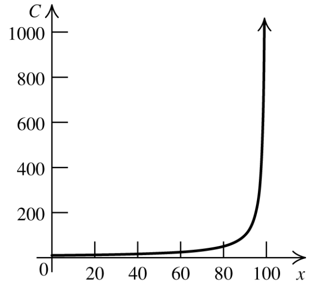
$C(75) = \dfrac{1000}{100-75} = 40$

$C(90) == \dfrac{1000}{100-90} = 100$

$C(99) = \dfrac{1000}{100-99} = 1000$

These show the estimated cost (in millions of dollars) of catching and convicting 50%, 75%, 90%, and 99% of the criminals, respectively.

b.

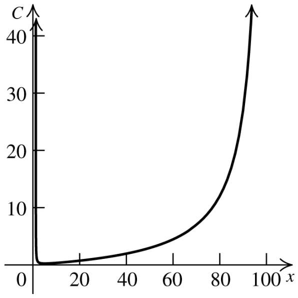


c. As $x \to 100^-, C(x) \to \infty.$

d. $30 = \dfrac{1000}{100 - x} \Rightarrow 3000 - 30x = 1000 \Rightarrow$
$2000 = 30x \Rightarrow x = 66.67\%$

89. a. $C(50) = \dfrac{3(50^2) + 50}{50(100 - 50)} = \$3.02$ billion

b.



c. $30 = \dfrac{3x^2 + 50}{x(100 - x)} = \dfrac{3x^2 + 50}{100x - x^2} \Rightarrow$
$3000x - 30x^2 = 3x^2 + 50 \Rightarrow$
$-33x^2 + 3000x - 50 = 0 \Rightarrow$
$x = \dfrac{-3000 \pm \sqrt{3000^2 - 4(-33)(-50)}}{2(-33)}$
$= \dfrac{-3000 \pm \sqrt{8,993,400}}{-66} \approx \dfrac{-3000 \pm 2998.9}{-66}$
≈ 90.89 or -0.017. Reject the negative solution. Approximately 90.89% of the impurities can be removed at a cost of \$30 billion.

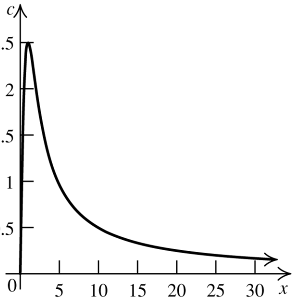
90. a. The horizontal asymptote is $y = a$. a is called the saturation level because as the concentration of the nutrient is increased, the growth rate is pushed close to a.

b. $g(k) = a\dfrac{k}{k + k} = a\dfrac{k}{2k} = \dfrac{a}{2}$

91. a. $P(0) = \dfrac{8(0) + 16}{2(0) + 1} = 16$ thousand $= 16,000$

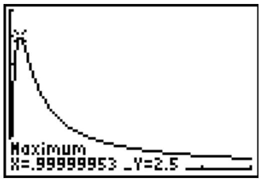
b. The horizontal asymptote is $y = 4$. This means that the population will stabilize at 4000.

92. a.



b. The horizontal asymptote is $y = 0$. This means that, as time passes, the concentration of the drug approaches 0.

c.



Maximum
X=.99999953 Y=2.5

The concentration of the drug in the bloodstream is maximal at 1 hour after the injection.

d. $2 = \dfrac{5t}{t^2 + 1} \Rightarrow 2t^2 - 5t + 2 = 0 \Rightarrow$
$(2t - 1)(x - 2) = 0 \Rightarrow t = \dfrac{1}{2}$ or $t = 2$

The concentration level equal 2 ml/l at $\frac{1}{2}$ hr and 2 hr after the injection.

93. a. $f(x) = \dfrac{10x + 200,000}{x - 2500}$

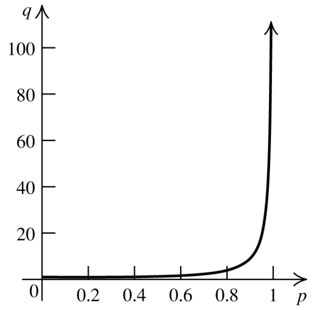
b. $C(10,000) = \dfrac{10(10,000) + 200,000}{10,000 - 2500} = \$40$

c. $20 > \dfrac{10x + 200,000}{x - 2500} \Rightarrow$
$20x - 50,000 > 10x + 200,000 \Rightarrow$
$10x > 250,000 \Rightarrow x > 25,000$
More than 25,000 books must be sold to bring the average cost under \$20.

d. The vertical asymptote is $x = 2500$. This represents the number of free samples. The horizontal asymptote is $y = 10$. This represents the cost of printing and binding one book.

94. a. $(1 + q)(1 - p) = 1 \Rightarrow 1 - p + q - pq = 1 \Rightarrow$
$q(1 - p) = p \Rightarrow q = \dfrac{p}{1 - p}$

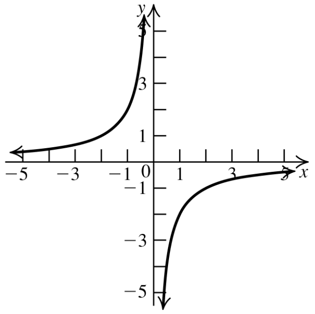
b.



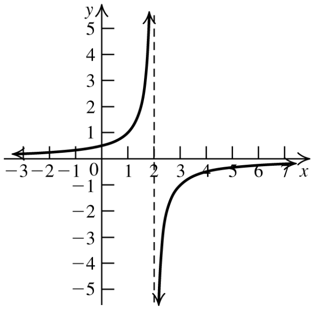
c. $q = \frac{0.25}{1-0.25} = 0.33 = 33.33\%$

3.6 Beyond the Basics

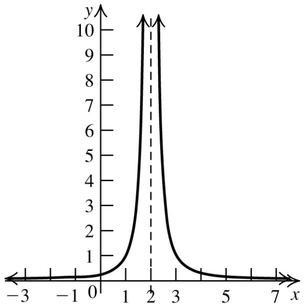
95. Stretch the graph of $y = \frac{1}{x}$ vertically by a factor of 2, and then reflect the graph about the x-axis.



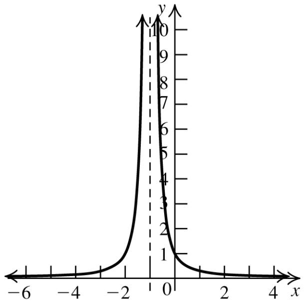
96. Shift the graph of $y = \frac{1}{x}$ two units right, and then reflect the graph about the x-axis.



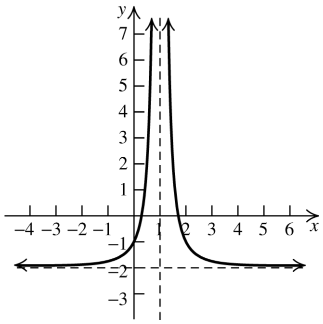
97. Shift the graph of $y = \frac{1}{x^2}$ two units right.



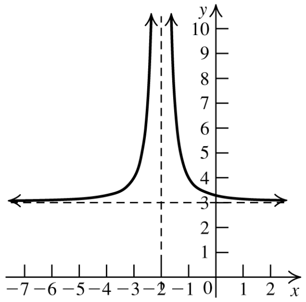
98. Shift the graph of $y = \frac{1}{x^2}$ one unit left.



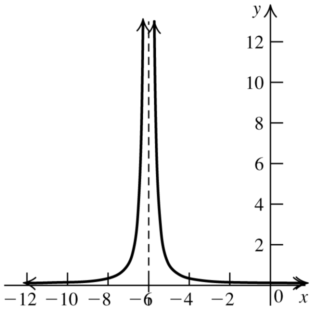
99. Shift the graph of $y = \frac{1}{x^2}$ one unit right and two units down.



100. Shift the graph of $y = \frac{1}{x^2}$ two units left and three units up.



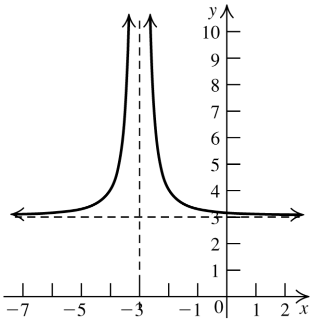
101. Shift the graph of $y=\frac{1}{x^2}$ six units left.



102. $$\begin{array}{r} 3 \\ x^2+6x+9\overline{\smash{\big)}3x^2+18x+28} \Rightarrow \\ \underline{3x^2+18x+27} \\ 1 \end{array}$$

$$\frac{3x^2+18x+28}{x^2+6x+9}=\frac{3(x^2+6x+9)+1}{x^2+6x+9}=$$
$$\frac{3(x^2+6x+9)}{x^2+6x+9}+\frac{1}{x^2+6x+9}=3+\frac{1}{(x+3)^2}$$

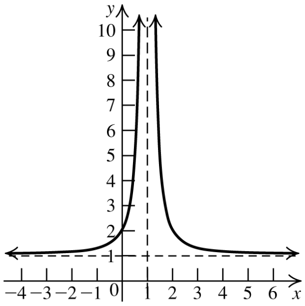
Shift the graph of $y=\frac{1}{x^2}$ three units left and three units up.



103. $$\begin{array}{r} 1 \\ x^2-2x+1\overline{\smash{\big)}x^2-2x+2} \Rightarrow \\ \underline{x^2-2x+1} \\ 1 \end{array}$$

$$\frac{x^2-2x+2}{x^2-2x+1}=\frac{(x^2-2x+1)+1}{x^2-2x+1}=$$
$$\frac{x^2-2x+1}{x^2-2x+1}+\frac{1}{x^2-2x+1}=1+\frac{1}{(x-1)^2}$$

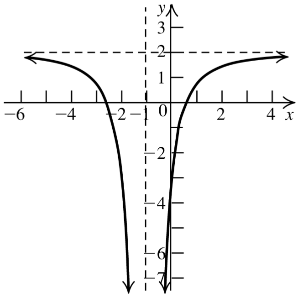
Shift the graph of $y=\frac{1}{x^2}$ one unit right and one unit up.



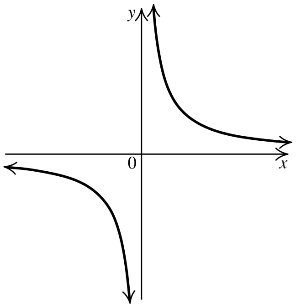
104. $$\begin{array}{r} 2 \\ x^2+2x+1\overline{\smash{\big)}2x^2+4x-3} \Rightarrow \\ \underline{2x^2+4x+2} \\ -5 \end{array}$$

$$\frac{2x^2+4x-3}{x^2+2x+1}=\frac{2(x^2+2x+1)-5}{x^2+2x+1}=$$
$$\frac{2(x^2+2x+1)}{x^2+2x+1}-\frac{5}{x^2+2x+1}=2-\frac{5}{(x+1)^2}$$

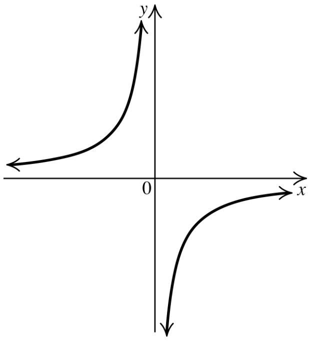
Shift the graph of $y=\frac{1}{x^2}$ one unit left, stretch the graph vertically by a factor of 5, reflect it about the *x*-axis, and shift it two units up.



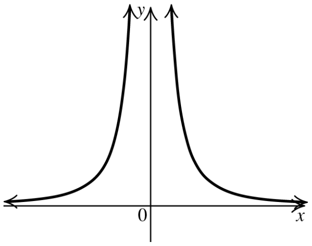
105. a. $f(x)\to 0$ as $x\to-\infty$; $f(x)\to 0$ as $x\to\infty$; $f(x)\to-\infty$ as $x\to 0^-$; $f(x)\to\infty$ as $x\to 0^+$. There are no *x*- or *y*-intercepts. The horizontal asymptote is the *x*-axis. The vertical asymptote is the *y*-axis.The graph is above the *x*-axis on $(0,\infty)$ and below the *x*-axis on $(-\infty,0)$.



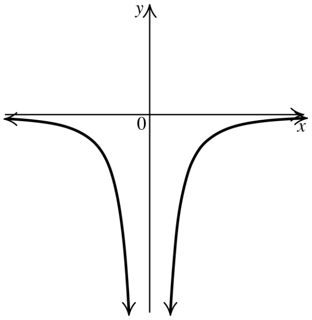
b. $f(x) \to 0$ as $x \to -\infty$; $f(x) \to 0$ as $x \to \infty$; $f(x) \to \infty$ as $x \to 0^-$; $f(x) \to -\infty$ as $x \to 0^+$. There are no *x*- or *y*-intercepts. The horizontal asymptote is the *x*-axis. The vertical asymptote is the *y*-axis. The graph is above the *x*-axis on $(-\infty, 0)$ and below the *x*-axis on $(0, \infty)$.



c. $f(x) \to 0$ as $x \to -\infty$; $f(x) \to 0$ as $x \to \infty$; $f(x) \to \infty$ as $x \to 0^-$; $f(x) \to \infty$ as $x \to 0^+$. There are no *x*- or *y*-intercepts. The horizontal asymptote is the *x*-axis. The vertical asymptote is the *y*-axis. The graph is above the *x*-axis on $(-\infty, 0) \cup (0, \infty)$.



d. $f(x) \to 0$ as $x \to -\infty$; $f(x) \to 0$ as $x \to \infty$; $f(x) \to -\infty$ as $x \to 0^-$; $f(x) \to -\infty$ as $x \to 0^+$. There are no *x*- or *y*-intercepts. The horizontal asymptote is the *x*-axis. The vertical asymptote is the *y*-axis. The graph is below the *x*-axis on $(-\infty, 0) \cup (0, \infty)$.



106. a. If *c* is a zero of *f*(*x*), then *x* = *c* is a vertical asymptote because *c* is not a factor of the numerator.

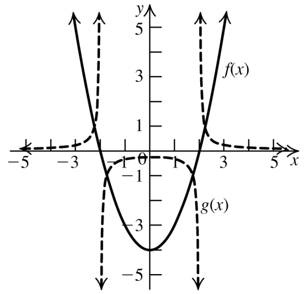
b. Because the numerators are positive, the signs of *f*(*x*) and *g*(*x*) are the same.

c. If the graphs intersect for some value *x*, then *f*(*x*) = *g*(*x*).

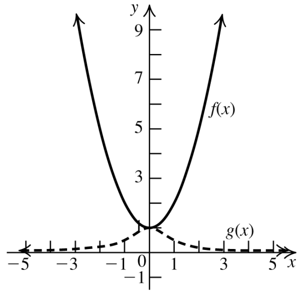
$f(x) = g(x) = \dfrac{1}{f(x)} \Rightarrow (f(x))^2 = 1 \Rightarrow$ $f(x) = \pm 1.$

d. If *f*(*x*) increases (decreases, remains constant), then the denominator of *g*(*x*) increases (decreases, remains constant). Therefore, *g*(*x*) decreases (increases, remains constant).

107.



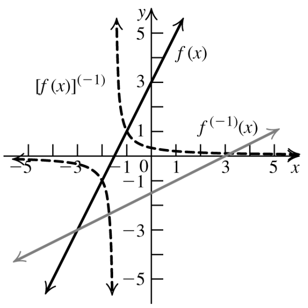
108.



109. $f(x) = 2x + 3 \Rightarrow [f(x)]^{-1} = \dfrac{1}{2x+3}$ and $y = 2x + 3$ becomes $x = 2y + 3 \Rightarrow$

$\dfrac{x-3}{2} = \dfrac{x}{2} - \dfrac{3}{2} = y = f^{-1}(x).$

Thus, the functions are different.



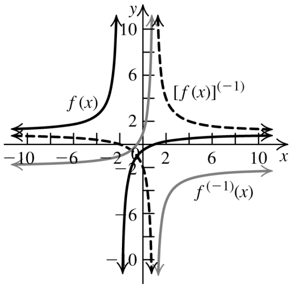
110. $f(x) = \dfrac{x-1}{x+2} \Rightarrow \left[f(x)\right]^{-1} = \dfrac{x+2}{x-1}$ and

$y = \dfrac{x-1}{x+2}$ becomes $x = \dfrac{y-1}{y+2} \Rightarrow$

$xy + 2x = y - 1 \Rightarrow xy - y = -2x - 1 \Rightarrow$

$y(x-1) = -2x - 1 \Rightarrow$

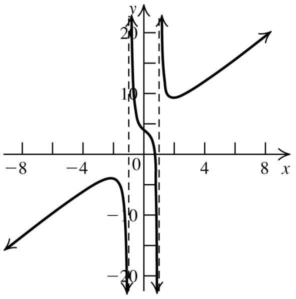
$y = \dfrac{-2x-1}{x-1} = \dfrac{2x+1}{1-x} = y = f^{-1}(x).$



111. $g(x) = \dfrac{2x^3 + 3x^2 + 2x - 4}{x^2 - 1} = 2x + 3 + \dfrac{-1}{x^2 - 1}$

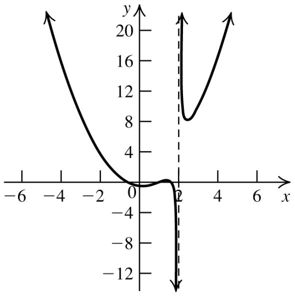
$g(x)$ has the oblique asymptote $y = 2x + 3$.

$g(x) \to -\infty$ as $x \to -\infty$, and $g(x) \to \infty$ as $x \to \infty$.



112. $f(x) = \dfrac{x^3 - 2x^2 + 1}{x - 2} = x^2 + \dfrac{1}{x - 2}$

$f(x)$ has no oblique asymptote. $f(x) \to \infty$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. For large $|x|$, the graph behaves like the graph of $y = x^2$.

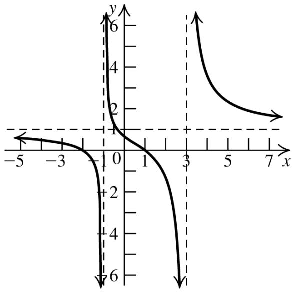


113. The horizontal asymptote is $y = 1$.

$\dfrac{x^2 + x - 2}{x^2 - 2x - 3} = 1 \Rightarrow x^2 + x - 2 = x^2 - 2x - 3 \Rightarrow$

$3x = -1 \Rightarrow x = -\dfrac{1}{3}.$

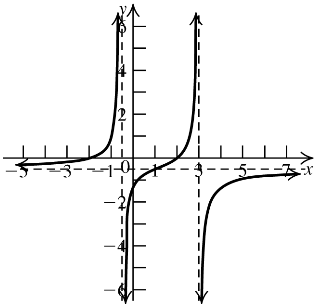
The point of intersection is $\left(-\dfrac{1}{3}, 1\right)$.



114. The horizontal asymptote is $y = -\dfrac{1}{2}$.

$\dfrac{4 - x^2}{2x^2 - 5x - 3} = -\dfrac{1}{2} \Rightarrow -8 + 2x^2 = 2x^2 - 5x - 3 \Rightarrow$

$x = 1$. The point of intersection is $\left(1, -\dfrac{1}{2}\right)$.



115. Vertical asymptote at $x = 3 \Rightarrow$ the denominator is $x - 3$. Horizontal asymptote at $y = -1 \Rightarrow$ the ratio of the leading coefficients of the numerator and denominator = −1. The *x*–intercept = $2 \Rightarrow$ the numerator is $x - 2$. So the equation is of the form $f(x) = -\dfrac{x-2}{x-3}$ or $\dfrac{2-x}{x-3}$.

116. Vertical asymptotes at $x = -1$ and $x = 1 \Rightarrow$ the denominator is $(x+1)(x-1)$. Horizontal asymptote at $y = 1 \Rightarrow$ the degrees of the numerator and denominator are the same. The *x*–intercept = $0 \Rightarrow$ the numerator is x^2. So the equation is of the form

$$f(x) = \frac{x^2}{(x-1)(x+1)} = \frac{x^2}{x^2 - 1}.$$

117. Vertical asymptote at $x = 0 \Rightarrow$ the denominator is x. The slant asymptote $y = -x \Rightarrow$ the degree of the numerator is one more than the degree of the denominator and the quotient of the numerator and the denominator is $-x$. The x-intercepts -1 and $1 \Rightarrow$ the numerator is $(x-1)(x+1)$. So the equation is of the form

$$f(x) = a\frac{(x-1)(x+1)}{x} = a\left(\frac{x^2-1}{x}\right).$$

$$\begin{array}{r} x \\ x\overline{)x^2-1} \\ \underline{x^2} \\ -1 \end{array}$$

However, the slant asymptote is $y = -x$, so $a = -1$ and the equation is

$$f(x) = -\frac{x^2-1}{x} = \frac{-x^2+1}{x}.$$

118. Vertical asymptote at $x = 3 \Rightarrow$ the denominator is $x - 3$. The slant asymptote $y = x + 4 \Rightarrow$ the degree of the numerator is one more than the degree of the denominator, and the quotient of the numerator and the denominator is $x + 4$. Thus, $(x - 3)(x + 4) + a$ is the numerator.

$$f(4) = 14 \Rightarrow f(x) = \frac{(x-3)(x+4)+a}{x-3} \Rightarrow$$

$$14 = \frac{4^2+4-12+a}{4-3} \Rightarrow a = 6$$

Therefore, the equation is

$$f(x) = \frac{(x-3)(x+4)+6}{x-3} = \frac{x^2+x-6}{x-3}$$

119. Vertical asymptote at $x = 2 \Rightarrow$ the denominator is $x - 2$. Horizontal asymptote at $y = 1 \Rightarrow$ the degrees of the numerator and denominator are the same and the leading coefficients of the numerator and denominator are the same. So the numerator is $x + a$, where a is chosen so that $f(0) = -2 \Rightarrow \dfrac{x+4}{x-2} = f(x).$

120. Vertical asymptotes at $x = 2$ and $x = -1 \Rightarrow$ the denominator is $(x-2)(x+1)$. Horizontal asymptote at $y = 0 \Rightarrow$ the degree of the numerator < the degree of the denominator. So the numerator is $x + a$, where a is chosen so that $f(0) = 2 \Rightarrow \dfrac{x-4}{(x-2)(x+1)} = f(x).$ Verify that the x-intercept is 4:

$$f(4) = \frac{4-4}{(4-2)(4+1)} = 0.$$

121. $f(x) \to -\infty$ as $x \to 1^-$ and $f(x) \to \infty$ as $x \to 1^+ \Rightarrow$ the denominator is zero if $x = 1$. Since $x \to 1$ from both directions, the denominator is $(x-1)^2$. $f(x) \to 4$ as $x \to \pm\infty \Rightarrow$ the horizontal asymptote is 4. So the leading coefficient of the numerator is 4 and the degree of the numerator is the same as the degree of the denominator. The numerator is $4x^2 + a$, where a is chosen so that

$$f\left(\frac{1}{2}\right) = 0 \Rightarrow \frac{4(1/2)^2+a}{(1/2-1)^2} \Rightarrow a = -1.\text{ So,}$$

$$f(x) = \frac{4x^2-1}{(x-1)^2}.$$

122. No vertical asymptotes $\Rightarrow$ the denominator has no real zeros. $f(x)$ symmetric about the y-axis $\Rightarrow f(x)$ is an even function. So the degree of the numerator and the degree of the denominator can equal 2. Then a possible denominator is $x^2 + 1$. $f(x) \to 2$ as $x \to \pm\infty \Rightarrow$ the horizontal asymptote is 2. So the numerator is $2x^2 + a$ where a is chosen so that $f(0) = 0 \Rightarrow a = 0$.

$$f(x) = \frac{2x^2}{x^2+1}.$$

123. Vertical asymptote at $x = 1 \Rightarrow$ the denominator could be $x - 1$. Because the oblique asymptote is $y = 3x + 2$, the numerator is $(3x+2)(x-1)+a$, where a can be any number (no x-intercepts or function values are given). Let $a = 1$. So.

$$f(x) = \frac{(3x+2)(x-1)+1}{x-1} = \frac{3x^2-x+1}{x-1}.$$

124. A rational function cannot have both a horizontal asymptote and an oblique asymptote because there would values of x which are mapped to two different function values. In that case, it wouldn't be a function.

3.6 Critical Thinking/Discussion/Writing

125. Answers may vary. Sample answers are given:

a. $f(x) = \dfrac{1}{x}$ **b.** $f(x) = \dfrac{x^2+2}{x^2+x-2}$

c. $f(x) = \dfrac{x^3+2x^2-7x-1}{x^3+x^2-6x+5}$

126. Answers may vary. Sample answers are given:

a. $f(x)=\dfrac{x^2+1}{x}$

b. $f(x)=\dfrac{x^2+2x-2}{x}$

c. $f(x)=\dfrac{x^5+x^4+x^2-4}{x^4}$

127. Since $y = x + 1$ is the oblique asymptote, we know that $R(x)=x+1+\dfrac{r(x)}{D(x)}$. We are told that the graph of $R(x)$ intersects the asymptote at the points $(2,\ R(2))$ and $(5,\ R(5))$, so $R(x)=x+1$ when $\dfrac{r(x)}{D(x)}=0.$ Thus, $r(x)=(x-2)(x-5).$ Recall that the numerator of a rational function must have degree greater than that of the denominator in order for there to be an oblique asymptote, so the denominator, $D(x)$, can be any polynomial whose degree is greater than or equal to 3. Thus, a possible rational function is

$$\begin{aligned}R(x)&=x+1+\frac{(x-2)(x-5)}{x^5}\\&=\frac{(x+1)x^5+(x-2)(x-5)}{x^5}.\end{aligned}$$

128. See exercise 127 for explanation.

$$\begin{aligned}R(x)&=(ax+b)+\frac{K(x-c_1)(x-c_2)\cdots(x-c_n)}{D(x)}\\&=\frac{(ax+b)+D(x)+K(x-c_1)(x-c_2)\cdots(x-c_n)}{D(x)},\end{aligned}$$

where $K\neq 0,\ D(x)$ is a polynomial of degree $n + 1$ and none of its zeros are at $c_1,\ c_2,\ \ldots,\ c_n$.

3.6 Maintaining Skills

129. a. $y-3=-\frac{2}{3}(x-(-5))\Rightarrow y-3=-\frac{2}{3}(x+5)\Rightarrow$
$y=-\frac{2}{3}x-\frac{2}{3}(5)+3\Rightarrow y=-\frac{2}{3}x-\frac{1}{3}$

b. $y=-\frac{2}{3}(1)-\frac{1}{3}=-1$

130. a. The slope of the line $4x+5y=6\Rightarrow$ $5y=-4x+6\Rightarrow y=-\frac{4}{5}x+\frac{6}{5}$ is $-\frac{4}{5}$. The slope of the line perpendicular to this line is $\frac{5}{4}$. The equation we are seeking is

$y-(-3)=\frac{5}{4}(x-(-2))\Rightarrow$

$y+3=\frac{5}{4}(x+2)\Rightarrow$

$y=\frac{5}{4}x+\frac{5}{4}(2)-3\Rightarrow y=\frac{5}{4}x-\frac{1}{2}.$

b. $y=\frac{5}{4}(10)-\frac{1}{2}=12$

131. $3(1)+2c=11\Rightarrow 2c=8\Rightarrow c=4$

132. $7=(-2)^2k+1\Rightarrow 7=4k+1\Rightarrow 6=4k\Rightarrow$
$k=\frac{3}{2}$

$y=\frac{3}{2}(2)^2+1=7$

133. $12=\dfrac{k}{\left(\frac{1}{2}\right)^2}\Rightarrow k=3$

$y=\frac{3}{2^2}\Rightarrow y=\frac{3}{4}$

3.7 Variation

3.7 Practice Problems

1. $y=kx\Rightarrow 6=30k\Rightarrow\frac{1}{5}=k$

$y=\left(\frac{1}{5}\right)120=24$

2. $I=kV\Rightarrow 60=220k\Rightarrow\frac{3}{11}=k$

$75=\left(\frac{3}{11}\right)V\Rightarrow 275=V$

A battery of 275 volts is needed to produce 60 amperes of current.

3. $y=kx^2\Rightarrow 48=k(2)^2\Rightarrow 12=k$
$y=12(5)^2=300$

4. $A = \frac{k}{B} \Rightarrow 12 = \frac{k}{5} \Rightarrow 60 = k$
$A = \frac{60}{3} = 20$

5. $y = \frac{k}{\sqrt{x}} \Rightarrow \frac{3}{4} = \frac{k}{\sqrt{16}} \Rightarrow k = 3$
$2 = \frac{3}{\sqrt{x}} \Rightarrow \sqrt{x} = \frac{3}{2} \Rightarrow x = \frac{9}{4}$

6. $F = G \cdot \frac{m_1 m_2}{r^2} \Rightarrow m \cdot g = G \cdot \frac{mM_{\text{Mars}}}{R_{\text{Mars}}^2} \Rightarrow$
$g = G \cdot \frac{M_{\text{Mars}}}{R_{\text{Mars}}^2}$

Note that the radius of Mars is given in kilometers, which must be converted to meters.

$$g = \frac{\left(6.67 \times 10^{-11} \text{ m}^3/\text{kg}/\text{sec}^2\right)\left(6.42 \times 10^{23} \text{ kg}\right)}{\left(3397 \text{ km}\right)^2}$$
$$= \frac{\left(6.67 \times 10^{-11} \text{ m}^3/\text{kg}/\text{sec}^2\right)\left(6.42 \times 10^{23} \text{ kg}\right)}{\left(3.397 \times 10^6 \text{ m}\right)^2}$$
$$\approx 3.7 \text{ m/sec}^2$$

3.7 Basic Concepts and Skills

1. $P = kT$

2. $P = \frac{k}{V}$

3. $y = k\sqrt{x}$

4. $F = km_1 m_2$

5. $V = kx^3 y^4$

6. $w = \frac{k\sqrt{x}}{y}$

7. $z = \frac{kxu}{v^2}$

8. $S = \frac{k\sqrt{x}y^2 z^3}{u^2}$

9. $x = ky; 15 = 30k \Rightarrow k = \frac{1}{2}; x = \frac{1}{2}(28) = 14$

10. $y = kx; 3 = 2k \Rightarrow k = \frac{3}{2}; y = \frac{3}{2}(7) = \frac{21}{2}$

11. $s = kt^2; 64 = 2^2 k \Rightarrow k = 16; s = 5^2(16) = 400$

12. $y = kx^3; 270 = 3^3 k \Rightarrow k = 10;$
$80 = 10x^3 \Rightarrow x = 2$

13. $r = \frac{k}{u}; 3 = \frac{k}{11} \Rightarrow k = 33; r = \frac{33}{1/3} = 99$

14. $y = \frac{k}{z}; 24 = \frac{k}{1/6} \Rightarrow k = 4; y = \frac{4}{1} = 4$

15. $B = \frac{k}{A^3}; 1 = \frac{k}{2^3} \Rightarrow k = 8; B = \frac{8}{4^3} = \frac{1}{8}$

16. $y = \frac{k}{\sqrt[3]{x}}; 10 = \frac{k}{\sqrt[3]{2}} \Rightarrow k = 10\sqrt[3]{2};$
$40 = \frac{10\sqrt[3]{2}}{\sqrt[3]{x}} \Rightarrow 40^3 = \frac{2000}{x} \Rightarrow x = \frac{1}{32}$

17. $z = kxy; 42 = (2)(3)k \Rightarrow k = 7;\ 56 = 2(7)y \Rightarrow$
$y = 4$

18. $m = \frac{kq}{p}; \frac{1}{2} = \frac{13k}{26} \Rightarrow k = 1; m = \frac{7(1)}{14} = \frac{1}{2}$

19. $z = kx^2; 32 = 4^2 k \Rightarrow k = 2; z = 2(5^2) = 50$

20. $u = \frac{k}{t^3}; 9 = \frac{k}{2^3} \Rightarrow k = 72; u = \frac{72}{6^3} = \frac{1}{3}$

21. $P = kTQ^2; 36 = (17)(6^2)k \Rightarrow k = \frac{1}{17};$
$P = \frac{1}{17}(4)(9^2) = \frac{324}{17}$

22. $a = kb\sqrt{c}; 9 = 13\sqrt{81}k \Rightarrow k = \frac{1}{13};$
$a = \frac{1}{13}(5)\sqrt{9} = \frac{15}{13}$

23. $z = \frac{k\sqrt{x}}{y^2}; 24 = \frac{\sqrt{16}k}{3^2} \Rightarrow k = 54$
$27 = \frac{54\sqrt{x}}{2^2} \Rightarrow 2 = \sqrt{x} \Rightarrow x = 4$

24. $z = \frac{kuv^3}{w^2}; 9 = \frac{4(3^3)k}{2^2} \Rightarrow k = \frac{1}{3};$
$8 = \frac{\frac{1}{3}(27)(2^3)}{w^2} \Rightarrow 8 = \frac{72}{w^2} \Rightarrow w = \pm 3$

25. $\frac{16}{12} = \frac{8}{y} \Rightarrow y = 6$

26. $\frac{17}{22} = \frac{z}{110} \Rightarrow z = 85$

27. $\frac{100}{x_0} = \frac{y}{2x_0} \Rightarrow y = 200$

28. $\frac{2}{\sqrt{9}} = \frac{3}{\sqrt{y}} \Rightarrow \sqrt{y} = \frac{9}{2} \Rightarrow y = \frac{81}{4}$

3.7 Applying the Concepts

29. $y = Hx$, where y is the speed of the galaxies and x is the distance between them.

30. $R = kP$, where k is a constant.

31. a. $y = 30.5x$, where y is the length in centimeters and x is the length in feet.

b. **(i)** $y = 30.5(8) = 244$ cm

(ii) $y = 30.5\left(5\frac{1}{3}\right) \approx 162.67$ cm

c. **(i)** $57 = 30.5x \Rightarrow x \approx 1.87$ ft

(ii) $124 = 30.5x \Rightarrow x \approx 4.07$ ft

32. a. $y = 2.2x$, where y is the weight in pounds and x is the weight in kilograms.

b. **(i)** $y = 2.2(0.125) = 0.275$ lb

(ii) $y = 2.2(4) = 8.8$ lb

(iii) $y = 2.2(2.4) = 5.28$ lb

c. **(i)** $27 = 2.2x \Rightarrow x \approx 12.27$ kg

(ii) $160 = 2.2x \Rightarrow x \approx 72.73$ kg

33. $P = kQ; 7 = 20k \Rightarrow \frac{7}{20} = k; P = \frac{7}{20}(100) = 35$ g

34. $W = kh; 600 = 40k \Rightarrow k = 15;$
$W = 15(25) = \$375.$
The constant of proportionality is the hourly wage.

35. $d = kt^2; 64 = 2^2k \Rightarrow k = 16; 9 = 16t^2 \Rightarrow$
$t = \frac{3}{4}$ sec

36. $F = kx; 10 = 4k \Rightarrow \frac{5}{2} = k; F = \frac{5}{2}(6) = 15$ lb

37. $P = \frac{k}{V}; 20 = \frac{k}{300} \Rightarrow k = 6000;$
$P = \frac{6000}{100} = 60 \text{ lb/in.}^2$

38. $P = \frac{kT_K}{V}; 36 = \frac{260k}{13} \Rightarrow k = \frac{9}{5}; P = \frac{9T_K}{5V}$

a. $40 = \frac{9(300)}{5V} \Rightarrow V = 13.5 \text{ in.}^3$

b. $P = \frac{9(280)}{5(39)} \approx 12.92 \text{ lb/in.}^2$

39. a. The astronaut is 6000 + 3960 miles from the Earth’s center.

$$W = \frac{k}{d^2}; 120 = \frac{k}{3960^2} \Rightarrow k = 120(3960^2)$$
$$W = \frac{120(3960^2)}{(6000+3960)^2} \approx 18.97 \text{ lb}$$

b. $$200 = \frac{k}{3960^2} \Rightarrow k = 200(3960^2)$$
$$W = \frac{200(3960^2)}{3950^2} \approx 201.01 \text{ lb}$$

40. You should buy at the higher altitude because an object weighs less the further it is from the Earth’s center.

41. 1740 km = 1,740,000 m = 1.740×10^6 m

$$g = G \cdot \frac{M}{R^2} = 6.67 \times 10^{-11} \times \frac{7.4 \times 10^{22}}{\left(1.740 \times 10^6\right)^2}$$
$$= \frac{(6.67)(7.4)(10^{11})}{1.740^2 \times 10^{12}} \approx 1.63 \text{ m/sec}^2$$

42. 696,000 km = 696,000,000 m = 6.96×10^8 m

$$g = G \bullet \frac{M}{R^2} = 6.67 \times 10^{-11} \times \frac{2 \times 10^{30}}{\left(6.96 \times 10^8\right)^2}$$
$$= \frac{(6.67)(2)(10^{19})}{6.96^2 \times 10^{16}} \approx 2.75 \times 10^2 \text{ m/sec}^2$$

43. a. $$I = \frac{k}{d^2}; 320 = \frac{k}{10^2} \Rightarrow k = 32{,}000$$
$$I = \frac{32{,}000}{5^2} = 1280 \text{ candlepower}$$

b. $$400 = \frac{32{,}000}{d^2} \Rightarrow d^2 = 80 \Rightarrow d \approx 8.94 \text{ ft}$$
from the source.

44. a. $s = k\sqrt{d}; 48 = k\sqrt{96} \Rightarrow k = 2\sqrt{6}$
$s = 2\sqrt{6d}$

b. **(i)** $s = 2\sqrt{6(60)} \approx 37.95$ mph

(ii) $s = 2\sqrt{6(150)} = 60$ mph

(iii) $s = 2\sqrt{6(200)} \approx 69.28$ mph

c. $70 = 2\sqrt{6d} \Rightarrow \sqrt{6d} = 35 \Rightarrow$
$6d = 35^2 = 1225 \Rightarrow d \approx 204.17$ ft

45. $p = k\sqrt{l} \Rightarrow k = \frac{p}{\sqrt{l}} = \frac{2p}{2\sqrt{l}} = \frac{2p}{\sqrt{4l}} \Rightarrow$ the length is multiplied by 4 if the period is doubled.

46. $V = kA; 400 = 100k \Rightarrow 4 = k$
$V = 4(120) = 480 \text{ cm}^3$

47. a. $H = kR^2N$, where k is a constant

b. $k(2R)^2N = 4kR^2N = 4H \Rightarrow$ the horsepower is multiplied by 4 if the radius is doubled.

c. $kR^2(2N) = 2kR^2N = 2H \Rightarrow$ the horsepower is doubled if the number of pistons is doubled.

d. $k\left(\frac{R}{2}\right)^2(2N) = \frac{2kR^2N}{4} = \frac{kR^2N}{2} = \frac{H}{2} \Rightarrow$ the horsepower is halved if the radius is halved and the number of pistons is doubled.

48. Convert the dimensions given in inches to feet.

$$s = \frac{kwd^2}{l}; 576 = \frac{k\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2}{25} \Rightarrow k = 172,800$$

$$s = \frac{172,800\left(\frac{1}{2}\right)\left(\frac{5}{6}\right)^2}{20} = 3000 \text{ lb}$$

3.7 Beyond the Basics

49. a. $E = kl^2v^3$

b. $1920 = k(10^2)(8^3) \Rightarrow k = \frac{3}{80} = 0.0375$

c. $E = 0.0375(8^2)(25^3) = 37,500$ watts

d. $kl^2(2v)^3 = 8kl^2v^3 = 8E$

e. $k(2l)^2v^3 = 4kl^2v^3 = 4E$

f. $k(2l)^2(2v)^3 = 4(8)kl^2v^3 = 32E$

50. a. $I_d = \frac{kI}{d^2}; 2 = \frac{100k}{2^2} \Rightarrow k = \frac{2}{25} = 0.08$

b. The distance from the bulb to the point on the floor is $\sqrt{2^2 + 3^2} = \sqrt{13}$ m.

$$I_d = \frac{0.08(200)}{\left(\sqrt{13}\right)^2} \approx 1.23 \text{ watts/m}^2$$

c. If the bulb is raised by 1 meter, then the distance from the bulb to the point on the floor is $\sqrt{3^2 + 3^2} = 3\sqrt{2}$ m.

$$I_d = \frac{0.08(200)}{\left(3\sqrt{2}\right)^2} \approx 0.89 \text{ watts/m}^2$$

51. a. $m = kw^{3/4}; 75 = k(75^{3/4}) \Rightarrow$
$k = \sqrt[4]{3}\sqrt{5} \approx 2.94$

b. $m = 2.94(450^{3/4}) \approx 287.25$ watts

c. $k(4w)^{3/4} = 4^{3/4}kw^{3/4} \approx 2.83kw^{3/4} \approx 2.83\text{m}$

d. $250 \approx 2.94w^{3/4} \Rightarrow w^{3/4} \approx 85.034 \Rightarrow$
$w \approx 373.93$ kg

52. $r_{\text{sun-Earth}} = 400r_{\text{moon-Earth}}$

$$F_{\text{s-E}} = 6.67 \times 10^{-11} \times \frac{m_{\text{sun}}m_{\text{Earth}}}{r^2_{\text{s-E}}}$$
$$= 6.67 \times 10^{-11} \times \frac{2 \times 10^{30} \times 6 \times 10^{24}}{r^2_{\text{s-E}}}$$
$$= \frac{8.004 \times 10^{44}}{r^2_{\text{s-E}}} = \frac{8.004 \times 10^{44}}{(400r_{\text{m-E}})^2}$$
$$= \frac{8.004 \times 10^{44}}{1.6 \times 10^5 r^2_{\text{m-E}}} = \frac{8.004 \times 10^{39}}{1.6r^2_{\text{m-E}}}$$

$$F_{\text{m-E}} = 6.67 \times 10^{-11} \times \frac{m_{\text{moon}}m_{\text{Earth}}}{r^2_{\text{m-E}}}$$
$$= 6.67 \times 10^{-11} \times \frac{7.4 \times 10^{22} \times 6 \times 10^{24}}{r^2_{\text{m-E}}}$$
$$= \frac{2.9615 \times 10^{37}}{r^2_{\text{m-E}}}$$

$$\frac{8.004 \times 10^{39}}{1.6r^2_{\text{m-E}}} \div \frac{2.9615 \times 10^{37}}{r^2_{\text{m-E}}}$$
$$= \frac{8.004 \times 10^{39}}{1.6r^2_{\text{m-E}}} \times \frac{r^2_{\text{m-E}}}{2.9615 \times 10^{37}} \approx 168.92$$

The gravitational attraction between the sun and the Earth is approximately 168.92 times as strong as the gravitational pull between the Earth and the moon.

53. a. $T^2 = \dfrac{4\pi^2 r^3}{G(M_1 + M_2)}$

b. Because gravity is in terms of cubic meters per kilogram per second squared, convert the distance from kilometers to meters:

$1.5 \times 10^8 \text{km} = 1.5 \times 10^{11} \text{m}$.

$$(3.15 \times 10^7)^2 \approx \frac{4\pi^2 (1.5 \times 10^{11})^3}{6.67 \times 10^{-11} M_{\text{sun}}} \Rightarrow$$

$$9.9225 \times 10^{14} \approx \frac{4\pi^2 (3.375) \times 10^{33}}{6.67 \times 10^{-11} M_{\text{sun}}} \Rightarrow$$

$$M_{\text{sun}} \approx \frac{133.24 \times 10^{24}}{6.67 \times 10^{-11} \times 9.9225 \times 10^{14}} \approx 2.01 \times 10^{30} \text{ kg}$$

54. Because gravity is in terms of cubic meters per kilogram per second squared, convert the distance from kilometers to meters:

$384{,}000 \text{km} = 3.84 \times 10^8 \text{m}$.

27.3 days $= 27.3(24)(60)(60)$ sec $= 2{,}358{,}720$ sec

$$2{,}358{,}720^2 = \frac{4\pi^2 (3.84 \times 10^8)^3}{6.67 \times 10^{-11} M_{\text{Earth}}} \Rightarrow$$

$$M_{\text{Earth}} = \frac{4\pi^2 (3.84 \times 10^8)^3}{6.67 \times 10^{-11} \times 2{,}358{,}720^2} \Rightarrow$$

$$M_{\text{Earth}} \approx 6.02 \times 10^{24} \text{ kg}$$

55. a. $R = kN(P - N)$, where k is the constant of proportionality.

b. $45 = 1000(9000)k \Rightarrow k = 5 \times 10^{-6}$

c. $R = 5 \times 10^{-6}(5000)(5000) = 125$ people per day.

d. $100 = 5 \times 10^{-6} N(10{,}000 - N) \Rightarrow$
$20{,}000{,}000 = 10{,}000N - N^2 \Rightarrow$
$N^2 - 10{,}000N + 20{,}000{,}000 = 0 \Rightarrow$

$$N = \frac{10{,}000 \pm \sqrt{10{,}000^2 - 4(1)(2 \times 10^7)}}{2(1)} = \frac{10{,}000 \pm \sqrt{20{,}000{,}000}}{2} \approx 2764 \text{ or } 7236$$

3.7 Critical Thinking/Discussion/Writing

56. $I = \dfrac{kV}{R} \Rightarrow IR = kV$

$1.3I = \dfrac{kV}{1.2R} \Rightarrow 1.56IR = kV \Rightarrow$ the voltage must increase by 56%.

57. a. $v = kw^2$. The diamond is cut into two pieces whose weights are $\dfrac{2w}{5}$ and $\dfrac{3w}{5}$. The value of the first piece is $\left(\dfrac{4}{25}\right)(1000) = \$160$, and the value of the second piece is $\left(\dfrac{9}{25}\right)(1000) = \$360$. The two pieces together are valued at \$520, a loss of \$480.

b. The stone is broken into three pieces whose weights are $\dfrac{5w}{25} = \dfrac{w}{5}, \dfrac{9w}{25},$ and $\dfrac{11w}{25}$. So, the values of the three pieces are

$\left(\dfrac{1}{5}\right)^2 (25{,}000) = \$1000,$

$\left(\dfrac{9}{25}\right)^2 (25{,}000) = \$3240,$ and

$\left(\dfrac{11}{25}\right)^2 (25{,}000) = \$4840,$ respectively. The total value is \$9,080, a loss of \$15,920.

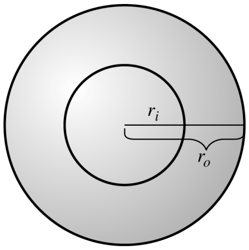
c. The weights of the pieces are $\dfrac{w}{15}, \dfrac{2w}{15}, \dfrac{3w}{15} = \dfrac{w}{5}, \dfrac{4w}{15},$ and $\dfrac{5w}{15} = \dfrac{w}{3},$ respectively. If the original value is x, then

$$x - 85{,}000 = \left(\frac{1}{15}\right)^2 x + \left(\frac{2}{15}\right)^2 x + \left(\frac{1}{5}\right)^2 x + \left(\frac{4}{15}\right)^2 x + \left(\frac{1}{3}\right)^2 x \Rightarrow$$

$x - 85{,}000 = \dfrac{11}{45} x \Rightarrow x = \$112{,}500 =$ the original value of the diamond. A diamond whose weight is twice that of the original diamond is worth 4 times the value of the original diamond = \$450,000.

58. $p = kd(n - f)$
$80 = k(40)(30 - f)$ and
$180 = k(60)(35 - f)$. Solving the first equation for k we have $k = -\frac{2}{f - 30}$.
Substitute that value into the second equation and solve for *f*:
$$180 = \left(\frac{2}{30 - f}\right)(60)(35 - f) \Rightarrow$$
$$180 = \frac{4200 - 120f}{30 - f} \Rightarrow$$
$$5400 - 180f = 4200 - 120f \Rightarrow$$
$$1200 = 60f \Rightarrow f = 20$$

59. $w_{\text{solid}} = k{r_o}^3; w_{\text{hollow}} = k{r_o}^3 - k{r_i}^3 = \frac{7}{8}k{r_o}^3$
$$k{r_o}^3 - k{r_i}^3 = \frac{7}{8}k{r_o}^3 \Rightarrow {r_o}^3 - {r_i}^3 = \frac{7}{8}{r_o}^3 \Rightarrow$$
$$\frac{1}{8}{r_o}^3 = {r_i}^3 \Rightarrow \frac{1}{8} = \frac{{r_i}^3}{{r_o}^3} \Rightarrow \frac{1}{2} = \frac{r_i}{r_o}$$



60. $s = 24 - k\sqrt{w}; 20 = 24 - k\sqrt{4} \Rightarrow k = 2$
$0 \le 24 - 2\sqrt{w} \Rightarrow w \le 144$
The greatest number of wagons the engine can move is 144.

3.7 Maintaining Skills

61. $5^0 = 1$

62. $2^3 = 8$

63. $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

64. $\left(\frac{1}{2}\right)^3 = \frac{1^3}{2^3} = \frac{1}{8}$

65. $\left(\frac{1}{2}\right)^{-4} = 2^4 = 16$

66. $2^{x-1} \cdot 2^{3-x} = 2^{(x-1)+(3-x)} = 2^2 = 4$

67. $5^{2x-3} \cdot 5^{3-x} = 5^{(2x-3)+(3-x)} = 5^x$

68. $\frac{2^{3x-2}}{2^{x-5}} = 2^{(3x-2)-(x-5)} = 2^{2x+3}$

69. $\left(2^{x-1}\right)^x = 2^{(x-1)(x)} = 2^{x^2-x}$

70. $\sqrt[3]{\sqrt[4]{2}} = \left(2^{1/4}\right)^{1/3} = 2^{(1/4)(1/3)} = 2^{1/12} = \sqrt[12]{2}$

71. $y = mx + b \Rightarrow y - b = mx \Rightarrow m = \frac{y - b}{x}$

72. $ax + by = 3 \Rightarrow ax = 3 - by \Rightarrow x = \frac{3 - by}{a}$

73. $A = B(1 + C) \Rightarrow \frac{A}{B} = 1 + C \Rightarrow \frac{A}{B} - 1 = C$

74. $A = B(1 + C)^3 \Rightarrow \frac{A}{B} = (1 + C)^3 \Rightarrow$
$\sqrt[3]{\frac{A}{B}} = 1 + C \Rightarrow \sqrt[3]{\frac{A}{B}} - 1 = C$

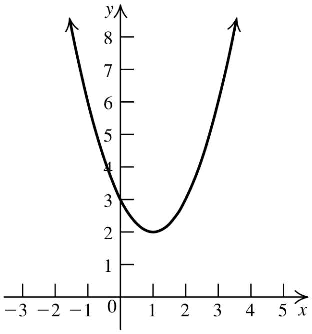
75. $A = B \cdot 10^{-n} \Rightarrow \frac{A}{10^{-n}} = B \Rightarrow A \cdot 10^n = B$

76. $A = B \cdot 10^m + C \Rightarrow A - C = B \cdot 10^m \Rightarrow$
$\frac{A - C}{10^m} = B$

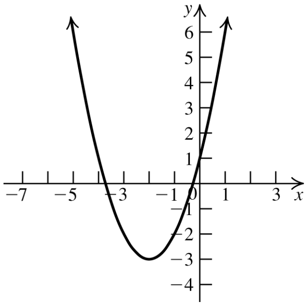
Chapter 3 Review Exercises

Basic Concepts and Skills

1. **(i)** Opens up **(ii)** Vertex: (1, 2)
(iii) Axis: $x = 1$
(iv) $0 = (x - 1)^2 + 2 \Rightarrow -2 = (x - 1)^2 \Rightarrow$ there are no *x*-intercepts.
(v) $y = (0 - 1)^2 + 2 \Rightarrow y = 3$ is the *y*-intercept.
(vi) The function is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.



2. **(i)** Opens up **(ii)** Vertex: (–2, –3)
(iii) Axis: $x = -2$



(iv) $0 = (x+2)^2 - 3 \Rightarrow 3 = (x+2)^2 \Rightarrow$ $x = -2 \pm \sqrt{3}$ are the x-intercepts.

(v) $y = (0+2)^2 - 3 \Rightarrow y = 1$ is the y-intercept.

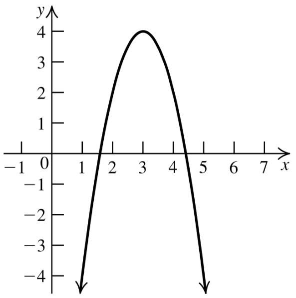
(vi) The function is decreasing on $(-\infty, -2)$ and increasing on $(-2, \infty)$.

3. **(i)** Opens down **(ii)** vertex: (3, 4)
(iii) Axis: $x = 3$

(iv) $0 = -2(x-3)^2 + 4 \Rightarrow 2 = (x-3)^2 \Rightarrow$ $x = 3 \pm \sqrt{2}$ are the x-intercepts.

(v) $y = -2(0-3)^2 + 4 \Rightarrow y = -14$ is the y-intercept.

(vi) The function is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$.

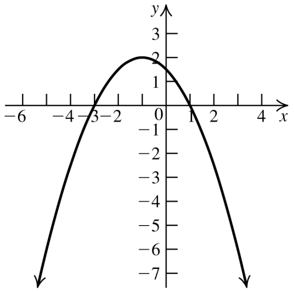


4. **(i)** Opens down **(ii)** Vertex: (–1, 2)
(iii) Axis: $x = -1$

(iv) $0 = -\frac{1}{2}(x+1)^2 + 2 \Rightarrow 4 = (x+1)^2 \Rightarrow$ $x = -3$ and $x = 1$ are the x-intercepts.

(v) $y = -\frac{1}{2}(0+1)^2 + 2 \Rightarrow y = \frac{3}{2}$ is the y-intercept.

(vi) The function is increasing on $(-\infty, -1)$ and decreasing on $(-1, \infty)$.

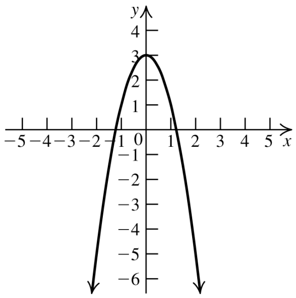


5. **(i)** Opens down **(ii)** Vertex: (0, 3)
(iii) Axis: $x = 0$ (y-axis)

(iv) $0 = -2x^2 + 3 \Rightarrow \frac{3}{2} = x^2 \Rightarrow$ $x = \pm\frac{\sqrt{6}}{2}$ are the x-intercepts.

(v) $y = -2(0)^2 + 3 \Rightarrow y = 3$ is the y-intercept.

(vi) The function is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.



6. **(i)** Opens up

(ii) To find the vertex, write the equation in standard form by completing the square:

$$y = 2x^2 + 4x - 1$$
$$y + 1 + 2 = 2(x^2 + 2x + 1)$$
$$y = 2(x+1)^2 - 3$$

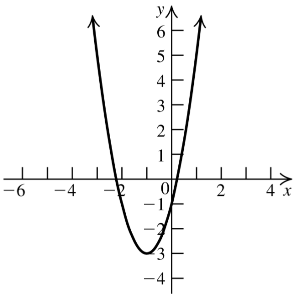
The vertex is (–1, –3).

(iii) Axis: $x = -1$

(iv) $0 = 2x^2 + 4x - 1 \Rightarrow x$ $= \frac{-4 \pm \sqrt{4^2 - 4(2)(-1)}}{2(2)} = \frac{-4 \pm \sqrt{24}}{4} \Rightarrow$ $x = -1 \pm \frac{\sqrt{6}}{2}$ are the x-intercepts.

(v) $y = 2(0)^2 + 4(0) - 1 \Rightarrow y = -1$ is the y-intercept.

(vi) The function is decreasing on $(-\infty, -1)$ and increasing on $(-1, \infty)$.



7. **(i)** Opens up

(ii) To find the vertex, write the equation in standard form by completing the square:

$$\begin{aligned} y &= 2x^2 - 4x + 3 \\ y - 3 + 2 &= 2(x^2 - 2x + 1) \\ y &= 2(x-1)^2 + 1 \end{aligned}$$

The vertex is (1, 1).

(iii) Axis: $x = 1$

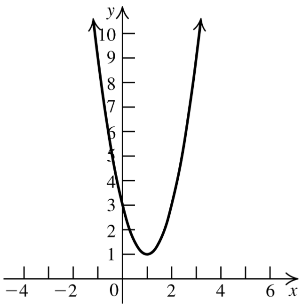
(iv) $0 = 2x^2 - 4x + 3 \Rightarrow$

$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(3)}}{2(2)} = \frac{4 \pm \sqrt{-8}}{4} \Rightarrow$$

there are no x-intercepts.

(v) $y = 2(0)^2 - 4(0) + 3 \Rightarrow y = 3$ is the y-intercept.

(vi) The function is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.



8. **(i)** Opens down

(ii) To find the vertex, write the equation in standard form by completing the square:

$$y = -2x^2 - x + 3 \Rightarrow$$

$$y - 3 - \frac{1}{8} = -2\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) \Rightarrow$$

$$y = -2\left(x + \frac{1}{4}\right)^2 + \frac{25}{8} \Rightarrow$$

the vertex is $\left(-\frac{1}{4}, \frac{25}{8}\right)$.

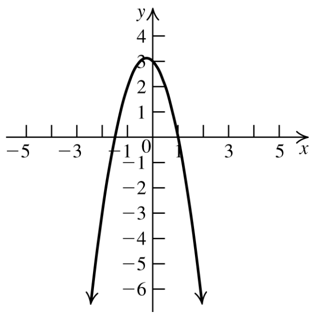
(iii) Axis: $x = -\frac{1}{4}$

(iv) $0 = -2x^2 - x + 3 \Rightarrow$

$0 = -(2x+3)(x-1) \Rightarrow x = -\frac{3}{2}$ and $x = 1$ are the x-intercepts.

(v) $y = -2(0)^2 - 0 + 3 \Rightarrow y = 3$ is the y-intercept.

(vi) The function is increasing on $\left(-\infty, -\frac{1}{4}\right)$ and decreasing on $\left(-\frac{1}{4}, \infty\right)$.



9. **(i)** Opens up

(ii) To find the vertex, write the equation in standard form by completing the square.

$$\begin{aligned} y &= 3x^2 - 2x + 1 \\ y - 1 + \frac{1}{3} &= 3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) \\ y &= 3\left(x - \frac{1}{3}\right)^2 + \frac{2}{3} \end{aligned}$$

The vertex is $\left(\frac{1}{3}, \frac{2}{3}\right)$.

(iii) Axis: $x = \frac{1}{3}$

(iv) $0 = 3x^2 - 2x + 1 \Rightarrow$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(3)(1)}}{2(3)} = \frac{2 \pm \sqrt{-8}}{6} \Rightarrow$$

there are no x-intercepts.

(v) $y = 3(0)^2 - 3(0) + 1 \Rightarrow y = 1$ is the y-intercept.

(vi) The function is decreasing on $\left(-\infty, \frac{1}{3}\right)$ and increasing on $\left(\frac{1}{3}, \infty\right)$.

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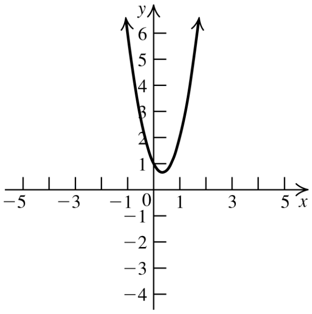


10. **(i)** Opens up

(ii) To find the vertex, write the equation in standard form by completing the square:

$$y = 3x^2 - 5x + 4$$
$$y - 4 + \frac{25}{12} = 3\left(x^2 - \frac{5}{3}x + \frac{25}{36}\right)$$
$$y = 3\left(x - \frac{5}{6}\right)^2 + \frac{23}{12}$$

The vertex is $\left(\frac{5}{6}, \frac{23}{12}\right)$.

(iii) Axis: $x = \frac{5}{6}$

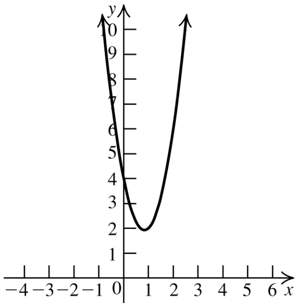
(iv) $0 = 3x^2 - 5x + 4 \Rightarrow$

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(3)(4)}}{2(3)} = \frac{5 \pm \sqrt{-23}}{6} \Rightarrow$$

there are no *x*-intercepts.

(v) $y = 3(0)^2 - 5(0) + 4 \Rightarrow y = 4$ is the *y*-intercept.

(vi) The function is decreasing on $\left(-\infty, \frac{5}{6}\right)$ and increasing on $\left(\frac{5}{6}, \infty\right)$.



In exercises 11–14, find the vertex using the formula $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

11. $a > 0 \Rightarrow$ the graph opens up, so *f* has a minimum value at its vertex. The vertex is

$$\left(-\frac{-4}{2(1)}, f\left(-\frac{-4}{2(1)}\right)\right) = (2, -1).$$

The minimum value is –1.

12. $a < 0 \Rightarrow$ the graph opens down, so *f* has a maximum value at its vertex. The vertex is

$$\left(-\frac{8}{2(-4)}, f\left(-\frac{8}{2(-4)}\right)\right) = (1, 1).$$

The maximum value is 1.

13. $a < 0 \Rightarrow$ the graph opens down, so *f* has a maximum value at its vertex. The vertex is

$$\left(-\frac{-3}{2(-2)}, f\left(-\frac{-3}{2(-2)}\right)\right) = \left(-\frac{3}{4}, \frac{25}{8}\right).$$

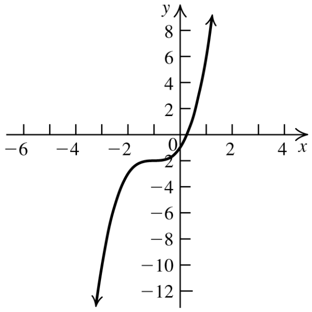
The maximum value is $\frac{25}{8}$.

14. $a > 0 \Rightarrow$ the graph opens up, so *f* has a minimum value at its vertex. The vertex is

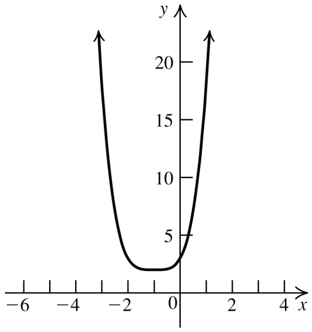
$$\left(-\frac{-3/4}{2(1/2)}, f\left(-\frac{-3/4}{2(1/2)}\right)\right) = \left(\frac{3}{4}, \frac{55}{32}\right).$$

The minimum value is $\frac{55}{32}$.

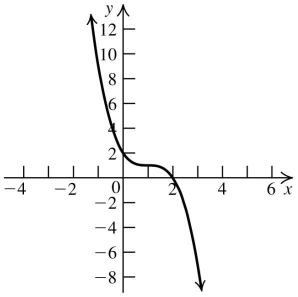
15. Shift the graph of $y = x^3$ one unit left and two units down.



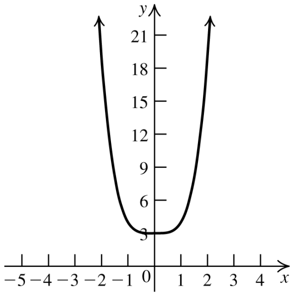
16. Shift the graph of $y = x^4$ one unit left and two units up.



17. Shift the graph of $y = x^3$ one unit right, reflect the resulting graph in the y-axis, and shift it one unit up.



18. Shift the graph of $y = x^4$ three units up.



19. **(i)** $f(x) \to -\infty$ as $x \to -\infty$
$f(x) \to \infty$ as $x \to \infty$

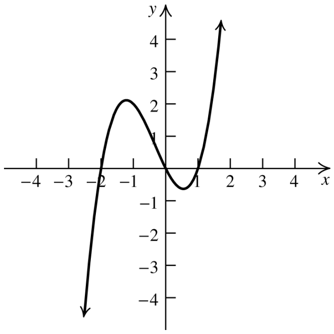
(ii) Zeros: $x = -2$, multiplicity 1, crosses the x-axis; $x = 0$, multiplicity 1, crosses the x-axis; $x = 1$, multiplicity 1, crosses the x-axis.

(iii) x-intercepts: –2, 0, 1;
$y = 0(0-1)(0+2) \Rightarrow y = 0$ is the y-intercept.

(iv) The intervals to be tested are $(-\infty,-2)$, $(-2,0)$, $(0,1), (1,2)$, and $(2,\infty)$.
The graph is above the x-axis on $(-2,0) \cup (1,\infty)$ and below the x-axis on $(-\infty,-2) \cup (0,1)$.

(v) $f(-x) = -x(-x-1)(-x+2) \neq f(x) \Rightarrow$
f is not even.
$-f(x) = -(x(x-1)(x+2)) \neq f(-x) \Rightarrow$
f is not odd. There are no symmetries.

(vi)



20. **(i)** $f(x) \to -\infty$ as $x \to -\infty$
$f(x) \to \infty$ as $x \to \infty$

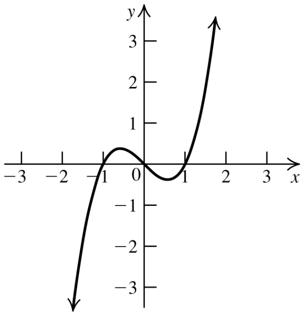
(ii) $f(x) = x^3 - x = x(x^2 - 1)$
$= x(x-1)(x+1)$
Zeros:
$x = -1$, multiplicity 1, crosses the x-axis;
$x = 0$, multiplicity 1, crosses the x-axis
$x = 1$, multiplicity 1, crosses the x-axis.

(iii) x-intercepts: –1, 0, 1;
$y = 0^3 - 0 \Rightarrow y = 0$ is the y-intercept.

(iv) The intervals to be tested are $(-\infty,-1)$, $(-1,0), (0,1)$, and $(1,\infty)$. The graph is above the x-axis on $(-1,0) \cup (1,\infty)$ and below the x-axis on $(-\infty,-1) \cup (0,1)$.

(v) $f(-x) = (-x)^3 - (-x) = -x^3 + x$
$= -f(x) \Rightarrow f$ is odd. f is symmetric with respect to the origin.

(vi)



21. **(i)** $f(x) \to -\infty$ as $x \to -\infty$
$f(x) \to -\infty$ as $x \to \infty$

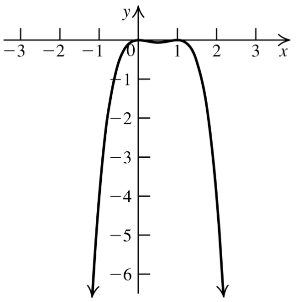
(ii) Zeros: $x = 0$, multiplicity 2, touches but does not cross the x- axis; $x = 1$, multiplicity 2, touches but does not cross the x- axis.

(iii) x-intercepts: 0, 1;
$y = -0^2(0-1)^2 \Rightarrow y = 0$ is the y-intercept.

(iv) The intervals to be tested are $(-\infty, 0)$, $(0,1)$, and $(1,\infty)$. The graph is below the x-axis on $(-\infty, 0) \cup (0,1) \cup (1,\infty)$.

(v) $f(-x) = -(-x)^2(-x-1)$
$= -x^2(-x-1) \ne -f(x)$ or $f(x) \Rightarrow$
$= -f(x) \Rightarrow f$ is neither even nor odd.
There are no symmetries.

(vi)



22. **(i)** $f(x) \to \infty$ as $x \to -\infty$
$f(x) \to -\infty$ as $x \to \infty$

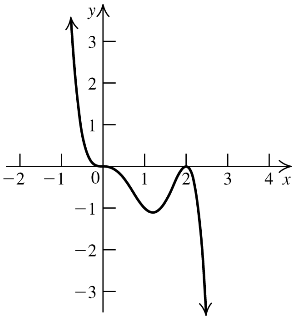
(ii) Zeros: $x = 0$, multiplicity 3, crosses the x-axis; $x = 2$, multiplicity 2, touches but does not cross the x-axis.

(iii) x-intercepts: 0, 2;
$y = -0^3(0-2)^2 \Rightarrow y = 0$ is the y-intercept.

(iv) The intervals to be tested are $(-\infty, 0)$, $(0,2)$, and $(2,\infty)$. The graph is above the x-axis on $(-\infty, 0)$ and below the x-axis on $(0,2) \cup (2,\infty)$.

(v) $f(-x) = -(-x)^3(-x-2)$
$= x^3(-x-2) \ne -f(x)$ or $f(x) \Rightarrow$
$= -f(x) \Rightarrow f$ is neither even nor odd.
There are no symmetries.

(vi)



23. **(i)** $f(x) \to -\infty$ as $x \to -\infty$
$f(x) \to -\infty$ as $x \to \infty$

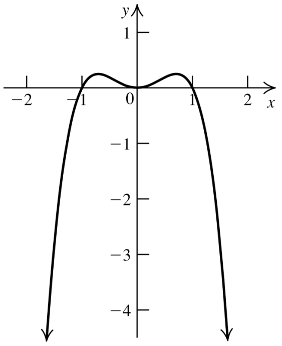
(ii) Zeros: $x = -1$, multiplicity 1, crosses the x-axis; $x = 0$, multiplicity 2, touches but does not cross; $x = 1$, multiplicity 1, crosses the x-axis.

(iii) x-intercepts: –1, 0, 1;
$y = -0^2(0^2-1) \Rightarrow y = 0$ is the y-intercept.

(iv) The intervals to be tested are $(-\infty, -1)$, $(-1,0), (0,1)$, and $(1,\infty)$. The graph is above the x-axis on $(-1,0) \cup (0,1)$ and below the x-axis on $(-\infty,-1) \cup (1,\infty)$.

(v) $f(-x) = -(-x)^2((-x)^2-1)$
$= -x^2(x^2-1) = f(x) \Rightarrow f$ is even, and f is symmetric with respect to the y-axis.

(vi)



24. **(i)** $f(x) \to -\infty$ as $x \to -\infty$
$f(x) \to -\infty$ as $x \to \infty$

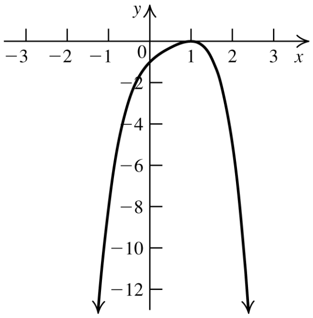
(ii) Zero: $x = 1$, multiplicity 2, touches but does not cross.

(iii) x-intercept: 1;
$y = -(0-1)^2(0^2+1) \Rightarrow y = -1$ is the y-intercept.

(iv) The intervals to be tested are $(-\infty, 1)$ and $(1,\infty)$. The graph is below the x-axis on $(-\infty,1) \cup (1,\infty)$.

(v) $f(-x) = -(-x-1)^2((-x)^2+1) \ne f(x)$ or $-f(x) \Rightarrow f$ is neither even nor odd.
There are no symmetries.

(vi)



25.
$$\begin{array}{r} 2x+3 \\ 3x-2\overline{\smash{\big)}6x^2+5x-13} \\ \underline{6x^2-4x} \\ 9x-13 \\ \underline{9x-\ \ 6} \\ -7 \end{array}$$

26.
$$\begin{array}{r} 4x-1 \\ 2x-3\overline{\smash{\big)}8x^2-14x+15} \\ \underline{8x^2-12x} \\ -2x+15 \\ \underline{-2x+\ \ 3} \\ 12 \end{array}$$

27.
$$\begin{array}{r} 8x^3-12x^2+14x-21 \\ x+1\overline{\smash{\big)}8x^4-4x^3+\ \ 2x^2-\ \ 7x+165} \\ \underline{8x^4+\ \ 8x^3} \\ -12x^3+\ \ 2x^2 \\ \underline{-12x^3-12x^2} \\ 14x^2-\ \ 7x \\ \underline{14x^2+14x} \\ -21x+165 \\ \underline{-21x-\ \ 21} \\ 186 \end{array}$$

28.
$$\begin{array}{r} x-1 \\ x^2-2x+6\overline{\smash{\big)}x^3-3x^2+4x+7} \\ \underline{x^3-2x^2+6x} \\ -x^2-2x+7 \\ \underline{-x^2+2x-6} \\ -4x+13 \end{array}$$

29.
$$\begin{array}{r|rrrr} 3 & 1 & 0 & -12 & 3 \\ & & 3 & 9 & -9 \\ \hline & 1 & 3 & -3 & -6 \end{array}$$

Quotient: x^2+3x-3 remainder –6.

30.
$$\begin{array}{r|rrrr} 6 & -4 & 3 & -5 & 0 \\ & & -24 & -126 & -786 \\ \hline & -4 & -21 & -131 & -786 \end{array}$$

Quotient: $-4x^2-21x-131$ remainder –786.

31.
$$\begin{array}{r|rrrrr} -1 & 2 & -3 & 5 & -7 & 165 \\ & & -2 & 5 & -10 & 17 \\ \hline & 2 & -5 & 10 & -17 & 182 \end{array}$$

Quotient: $2x^3-5x^2+10x-17$
remainder 182.

32.
$$\begin{array}{r|rrrrrr} -2 & 3 & -2 & 0 & 1 & -16 & -132 \\ & & -6 & 16 & -32 & 62 & -92 \\ \hline & 3 & -8 & 16 & -31 & 46 & -224 \end{array}$$

Quotient: $3x^4-8x^3+16x^2-31x+46$
remainder –224.

33. (i) $f(2)=2^3-3(2^2)+11(2)-29=-11$

(ii)
$$\begin{array}{r|rrrr} 2 & 1 & -3 & 11 & -29 \\ & & 2 & -2 & 18 \\ \hline & 1 & -1 & 9 & -11 \end{array}$$

The remainder is –11, so *f*(2) = –11.

34. (i) $f(-2)=2(-2)^3+(-2)^2-15(-2)-2=16$

(ii)
$$\begin{array}{r|rrrr} -2 & 2 & 1 & -15 & -2 \\ & & -4 & 6 & 18 \\ \hline & 2 & -3 & -9 & 16 \end{array}$$

The remainder is 16, so *f*(–2) = 16.

35. (i) $f(2)=(-3)^4-2(-3)^2-5(-3)+10=88$

(ii)
$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -2 & -5 & 10 \\ & & -3 & 9 & -21 & 78 \\ \hline & 1 & -3 & 7 & -26 & 88 \end{array}$$

The remainder is 88, so *f*(–3) = 88.

36. (i) $f(1)=1^5+2=3$

(ii)
$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 0 & 0 & 0 & 2 \\ & & 1 & 1 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 3 \end{array}$$

The remainder is 3, so *f*(1) = 3.

37.
$$\begin{array}{r|rrrr} 2 & 1 & -7 & 14 & -8 \\ & & 2 & -10 & 8 \\ \hline & 1 & -5 & 4 & 0 \end{array}$$

The remainder is 0, so 2 is a zero. Now find the zeros of the depressed function $x^2-5x+4=(x-4)(x-1)\Rightarrow$ 4 and 1 are also zeros. So the zeros of $x^3-7x^2+14x-8$ are 1, 2, and 4.

38. $$\begin{array}{r|rrrr} -2 & 2 & -3 & -12 & 4 \\ & & -4 & 14 & -4 \\ \hline & 2 & -7 & 2 & 0 \end{array}$$

The remainder is 0, so –2 is a zero. Now find the zeros of the depressed function $2x^2 - 7x + 2$:

$$x = \frac{7 \pm \sqrt{(-7)^2 - 4(2)(2)}}{2(2)} = \frac{7 \pm \sqrt{33}}{4}.$$

So the zeros of $2x^3 - 3x^2 - 12x + 4$ are –2 and $\frac{7 \pm \sqrt{33}}{4}$.

39. $$\begin{array}{r|rrrr} \frac{1}{3} & 3 & 14 & 13 & -6 \\ & & 1 & 5 & 6 \\ \hline & 3 & 15 & 18 & 0 \end{array}$$

The remainder is 0, so $1/3$ is a zero. Now find the zeros of the depressed function $3x^2 + 15x + 18 = 3(x + 2)(x + 3) \Rightarrow$ –2 and –3 are also zeros. So the zeros of $3x^3 + 14x^2 + 13x - 6$ are –3, –2, and $1/3$.

40. $$\begin{array}{r|rrrr} \frac{1}{4} & 4 & 19 & -13 & 2 \\ & & 1 & 5 & -2 \\ \hline & 4 & 20 & -8 & 0 \end{array}$$

The remainder is 0, so $1/4$ is a zero. Now find the zeros of the depressed function

$$4x^2 + 20x - 8 = \frac{-20 \pm \sqrt{20^2 - 4(4)(-8)}}{2(4)}$$
$$= \frac{-20 \pm 4\sqrt{33}}{8} = \frac{-5 \pm \sqrt{33}}{2}.$$ So the zeros of $4x^3 + 19x^2 - 13x + 2$ are $\frac{-5 \pm \sqrt{33}}{2}$ and $\frac{1}{4}$.

41. The factors of the constant term are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$, and the factors of the leading coefficient are $\{\pm 1\}$. So the possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$.

42. The factors of the constant term are $\{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$, and the factors of the leading coefficient are $\{\pm 1, \pm 3, \pm 9\}$. The possible rational zeros are $\Big\{\pm\frac{1}{9}, \pm\frac{2}{9}, \pm\frac{1}{3},$ $\pm\frac{4}{9}, \pm\frac{2}{3}, \pm\frac{8}{9}, \pm 1, \pm\frac{4}{3}, \pm\frac{16}{9}, \pm 2, \pm\frac{8}{3}, \pm 4,$ $\pm\frac{16}{3}, \pm 8, \pm 16\Big\}.$

43. $$\begin{aligned} f(x) &= 5x^3 + 11x^2 + 2x; \\ f(-x) &= 5(-x)^3 + 11(-x)^2 + 2(-x) \\ &= -5x^3 + 11x^2 - 2x \end{aligned}$$

There are no sign changes in $f(x)$, so there are no positive zeros. There are two sign changes in $f(-x)$, so there are 2 or 0 negative zeros.

$5x^3 + 11x^2 + 2x = x(5x^2 + 11x + 2)$
$= x(5x + 1)(2 + x) \Rightarrow$ the zeros are $-2, -\frac{1}{5}, 0.$

44. $$\begin{aligned} f(x) &= x^3 + 2x^2 - 5x - 6 \\ f(-x) &= (-x)^3 + 2(-x)^2 - 5(-x) - 6 \\ &= -x^3 + 2x^2 + 5x - 6 \end{aligned}$$

There is one sign change in $f(x)$, so there is one positive zero. There are two sign changes in $f(-x)$, so there are 2 or 0 negative zeros. The possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. Use synthetic division to find one zero:

$$\begin{array}{r|rrrr} -3 & 1 & 2 & -5 & -6 \\ & & -3 & 3 & 6 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

The remainder is 0, so –3 is a zero. Now find the zeros of the depressed function.
$x^2 - x - 2 = (x - 2)(x + 1) \Rightarrow$ –1 and 2 are also zeros. So the zeros of $x^3 + 2x^2 - 5x - 6$ are –3, –1, and 2.

45. $$\begin{aligned} f(x) &= x^3 + 3x^2 - 4x - 12 \\ f(-x) &= (-x)^3 + 3(-x)^2 - 4(-x) - 12 \\ &= -x^3 + 3x^2 + 4x - 12 \end{aligned}$$

There is one sign change in $f(x)$, so there is one positive zero. There are two sign changes in $f(-x)$, so there are 2 or 0 negative zeros.

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The possible rational zeros are $\{\pm1, \pm2, \pm3, \pm4, \pm6, \pm12\}$. Use synthetic division to find one zero:

$$\begin{array}{r|rrrr} -3 & 1 & 3 & -4 & -12 \\ & & -3 & 0 & 12 \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

The remainder is 0, so –3 is a zero. Now find the zeros of the depressed function $x^2 - 4 = (x-2)(x+2) \Rightarrow$ –2 and 2 are also zeros. So the zeros of $x^3 + 3x^2 - 4x - 12$ are –3, –2, and 2.

46. $f(x) = 2x^3 - 9x^2 + 12x - 5;$

$$f(-x) = 2(-x)^3 - 9(-x)^2 + 12(-x) - 5$$
$$= -2x^3 - 9x^2 - 12x - 5$$

There are three sign changes in $f(x)$, so there are 3 or 1 positive zeros. There are no sign changes in $f(-x)$, so there are no negative zeros. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm1, \pm\frac{5}{2}, \pm5\right\}$. Use synthetic division to find one zero:

$$\begin{array}{r|rrrr} 1 & 2 & -9 & 12 & -5 \\ & & 2 & -7 & 5 \\ \hline & 2 & -7 & 5 & 0 \end{array}$$

The remainder is 0, so 1 is a zero. Now find the zeros of the depressed function $2x^2 - 7x + 5 \Rightarrow (2x-5)(x-1) = 0 \Rightarrow$ $x = 5/2$ and 1 are also zeros. So, the zeros of $2x^3 - 9x^2 + 12x - 5$ are $5/2$ and 1 (multiplicity 2).

47. $f(x) = x^3 - 4x^2 - 5x + 14$

$$f(-x) = (-x)^3 - 4(-x)^2 - 5(-x) + 14$$
$$= -x^3 - 4x^2 + 5x + 14$$

There are two sign changes in $f(x)$, so there are 0 or 2 real positive zeros. There is one sign change in $f(-x)$, so there is one real negative zero. The possible rational zeros are $\{\pm1, \pm2, \pm7, \pm14\}$. Use synthetic division to find one zero:

$$\begin{array}{r|rrrr} -2 & 1 & -4 & -5 & 14 \\ & & -2 & 12 & -14 \\ \hline & 1 & -6 & 7 & 0 \end{array}$$

The remainder is 0, so –2 is a zero. Now find the zeros of the depressed function $x^2 - 6x + 7$.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2} = 3 \pm \sqrt{2}$$

So, the zeros of $x^3 - 4x^2 - 5x + 14$ are –2, $3 - \sqrt{2}$, and $3 + \sqrt{2}$.

48. $f(x) = x^3 - 5x^2 + 3x + 1$

$$f(-x) = (-x)^3 - 5(-x)^2 + 3(-x) + 1$$
$$= -x^3 - 5x^2 - 3x + 1$$

There are two sign changes in $f(x)$, so there are 0 or 2 real positive zeros. There is one sign change in $f(-x)$, so there is one real negative zero. The possible rational zeros are $\{\pm1\}$. Use synthetic division to find one zero:

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 3 & 1 \\ & & 1 & -4 & -1 \\ \hline & 1 & -4 & -1 & 0 \end{array}$$

The remainder is 0, so 1 is a zero. Now find the zeros of the depressed function $x^2 - 4x - 1$.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

So, the zeros of $x^3 - 5x^2 + 3x + 1$ are 1, $2 - \sqrt{5}$, and $2 + \sqrt{5}$.

49. $f(x) = 2x^3 - 5x^2 - 2x + 2$

$$f(-x) = 2(-x)^3 - 5(-x)^2 - 2(-x) + 2$$
$$= -2x^3 - 5x^2 + 2x + 2$$

There are two sign changes in $f(x)$, so there are 0 or 2 real positive zeros. There is one sign change in $f(-x)$, so there is one real negative zero. The possible rational zeros are $\left\{\pm1, \pm2, \pm\frac{1}{2}\right\}$. Use synthetic division to find one zero:

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & -2 & 2 \\ & & 1 & -2 & -2 \\ \hline & 2 & -4 & -4 & 0 \end{array}$$

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The remainder is 0, so $1/2$ is a zero. Now find the zeros of the depressed function $2x^2-4x-4.$

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(-4)}}{2(2)}$$
$$=\frac{4\pm\sqrt{48}}{4}=\frac{4\pm4\sqrt{3}}{4}=1\pm\sqrt{3}$$

So, the zeros of $2x^3-4x^2-2x+2$ are $1/2$, $1-\sqrt{3}$, and $1+\sqrt{3}$.

50. $f(x)=3x^3-29x^2+29x+13$
$$f(-x)=3(-x)^3-29(-x)^2+29(-x)+13$$
$$=-3x^3-29x^2-29x+13$$
There are two sign changes in $f(x)$, so there are 0 or 2 real positive zeros. There is one sign change in $f(-x)$, so there is one real negative zero. The possible rational zeros are $\left\{\pm1,\ \pm13,\ \pm\frac{1}{3},\ \pm\frac{13}{3}\right\}$. Use synthetic division to find one zero:

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & -29 & 29 & 13 \\ & & -1 & 10 & -13 \\ \hline & 3 & -30 & 39 & 0 \end{array}$$

The remainder is 0, so $-1/3$ is a zero. Now find the zeros of the depressed function $3x^2-30x+39=3\left(x^2-10x+13\right).$

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(13)}}{2(1)}$$
$$=\frac{10\pm\sqrt{48}}{2}=\frac{10\pm4\sqrt{3}}{2}=5\pm2\sqrt{3}$$

So, the zeros of $3x^3-29x^2+29x+13$ are $-1/3$, $5-2\sqrt{3}$, and $5+2\sqrt{3}$.

51. The function has degree three, so there are three zeros. Use synthetic division to find the depressed function:

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -7 & 6 \\ & & 2 & 4 & -6 \\ \hline & 1 & 2 & -3 & 0 \end{array}$$

Now find the zeros of the depressed function $x^2+2x-3=(x+3)(x-1)\Rightarrow -3$ and 1 are zeros. The zeros of the original function are –3, 1, 2.

52. The function has degree four, so there are four zeros. Since 1 is a zero of multiplicity 2, $(x-1)^2$ is a factor of $x^4+x^3-3x^2-x+2.$ Use synthetic division twice to find the depressed function:

$$\begin{array}{r|rrrrr} 1 & 1 & 1 & -3 & -1 & 2 \\ & & 1 & 2 & -1 & -2 \\ \hline & 1 & 2 & -1 & -2 & 0 \end{array} \qquad \begin{array}{r|rrrr} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

Alternatively, divide $x^4+x^3-3x^2-x+2$ by $(x-1)^2=x^2-2x+1$. Now find the zeros of the depressed function $x^2+3x+2=(x+2)(x+1)\Rightarrow -2$ and -1 are zeros. The zeros of the original function are –2, –1, and 1.

53. The function has degree four, so there are four zeros. Use synthetic division twice to find the depressed function:

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & 6 & -18 & -27 \\ & & -1 & 3 & -9 & 27 \\ \hline & 1 & -3 & 9 & -27 & 0 \end{array} \qquad \begin{array}{r|rrrr} 3 & 1 & -3 & 9 & -27 \\ & & 3 & 0 & 27 \\ \hline & 1 & 0 & 9 & 0 \end{array}$$

Alternatively, divide $x^4-2x^3+6x^2-18x-27$ by $(x+1)(x-3)=x^2-2x-3.$

Now find the zeros of the depressed function $x^2+9\Rightarrow \pm3i$ are zeros. The zeros of the original function are –1, 3, and $\pm3i$.

54. The function has degree three, so there are three zeros. Since one zero is 2 – *i*, another zero is 2 + *i*. Divide $4x^3-19x^2+32x-15$ by $(x-(2-i))(x-(2+i))=x^2-4x+5$ to find the depressed function:

$$\begin{array}{r} 4x-3 \\ x^2-4x+5\overline{\smash{)}4x^3-19x^2+32x-15} \\ \underline{4x^3-16x^2+20x} \\ -3x^2+12x-15 \\ \underline{-3x^2+12x-15} \\ 0 \end{array}$$

Now find the zeros of the depressed function: $4x-3=0\Rightarrow x=\frac{3}{4}.$ Alternatively, find the possible rational zeros, and then use synthetic division to find the zero. The zeros of the original function are $\frac{3}{4}, 2-i,$ and $2+i.$

55. The function has degree four, so there are four zeros. Since one zero is –1 + 2*i*, another zero is –1 – 2*i*. Divide $x^4+2x^3+9x^2+8x+20$ by $(x-(-1+2i))(x-(-1-2i)) = x^2+2x+5$ to find the depressed function:

$$\begin{array}{r} x^2+4 \\ x^2+2x+5\overline{\smash{)}x^4+2x^3+9x^2+8x+20} \\ \underline{x^4+2x^3+5x^2} \qquad\qquad \\ 4x^2+8x+20 \\ \underline{4x^2+8x+20} \\ 0 \end{array}$$

Now find the zeros of the depressed function: $x^2+4=0 \Rightarrow x=\pm 2i.$ The zeros of the original function are $\pm 2i$ and $-1\pm 2i.$

56. The function has degree five, so there are five zeros. Since two zeros are 2 + 2*i*, 2 – 2*i* are also zeros. Divide $x^5-7x^4+24x^3-32x^2+64$ by $\left((x-(2+2i))(x-(2-2i))\right)^2 = (x^2-4x+8)^2$ $= x^4-8x^3+32x^2-64x+64$ to find the depressed function. (Note it may be easier to divide by x^2-4x+8 and then to divide the quotient by x^2-4x+8.) The quotient is $x+1,$ so –1 is a zero. The zeros of the original function are -1 and $2\pm 2i.$

57. The function has degree three, so there are three zeros. The possible rational zeros are $\{\pm 1, \pm 2, \pm 4\}$. Using synthetic division, we find that one zero is 1:

$$\begin{array}{r|rrrr} \underline{1} & 1 & -1 & -4 & 4 \\ & & 1 & 0 & -4 \\ \hline & 1 & 0 & -4 & 0 \end{array}$$

Now find the zeros of the depressed function: $x^2-4=0 \Rightarrow x=\pm 2.$ The solution set is {–2, 1, 2}.

58. The function has degree three, so there are three zeros. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2, \pm 3, \pm 6\right\}$. Using synthetic division, we find that one zero is $-1/2$:

$$\begin{array}{r|rrrr} \underline{-1/2} & 2 & 1 & -12 & -6 \\ & & -1 & 0 & 6 \\ \hline & 2 & 0 & -12 & 0 \end{array}$$

Now find the zeros of the depressed function: $2x^2-12=0 \Rightarrow x=\pm\sqrt{6}.$ The solution set is $\left\{-1/2, \pm\sqrt{6}\right\}$.

59. The function has degree three, so there are three zeros. The possible rational zeros are $\left\{\pm\frac{1}{4}, \pm\frac{1}{2}, \pm\frac{3}{4}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$. Using synthetic division, we find that one zero is –1:

$$\begin{array}{r|rrrr} \underline{-1} & 4 & 0 & -7 & -3 \\ & & -4 & 4 & 3 \\ \hline & 4 & -4 & -3 & 0 \end{array}$$

Now find the zeros of the depressed function: $4x^2-4x-3=0 \Rightarrow (2x-3)(2x+1)=0 \Rightarrow$ $x=\frac{3}{2}$ or $x=-\frac{1}{2}.$ The solution set is $\left\{-1/2, -1, 3/2\right\}$.

60. The function has degree three, so there are three zeros. The possible rational zeros are $\{\pm 1, \pm 2, \pm 3, \pm 6\}$. Using synthetic division, we find that one zero is –3:

$$\begin{array}{r|rrrr} \underline{-3} & 1 & 1 & -8 & -6 \\ & & -3 & 6 & 6 \\ \hline & 1 & -2 & -2 & 0 \end{array}$$

Now find the zeros of the depressed function:

$$x^2-2x-2=0 \Rightarrow x=\frac{2\pm\sqrt{(-2)^2-4(1)(-2)}}{2(1)}$$

$$=\frac{2\pm\sqrt{12}}{2}=1\pm\sqrt{3}.$$

The solution set is $\left\{-3, 1\pm\sqrt{3}\right\}$.

61. $x^3-8x^2+23x-22=0$

The function has degree three, so there are three zeros. The possible rational zeros are $\{\pm 1, \pm 2, \pm 11, \pm 22\}$. Using synthetic division, we find that one zero is 2:

$$\begin{array}{r|rrrr} \underline{2} & 1 & -8 & 23 & -22 \\ & & 2 & -12 & 22 \\ \hline & 1 & -6 & 11 & 0 \end{array}$$

Now find the zeros of the depressed function: $x^2-6x+11=0.$

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(11)}}{2(1)}$$

$$=\frac{6\pm\sqrt{-8}}{2}=\frac{6\pm 2i\sqrt{2}}{2}=3\pm i\sqrt{2}$$

Solution set: $\left\{2, 3\pm i\sqrt{2}\right\}$

62. $x^3-3x^2+8x+12=0$
The function has degree three, so there are three zeros. The possible rational zeros are $\{\pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12\}$. Using synthetic division, we find that one zero is -1:

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 8 & 12 \\ & & -1 & 4 & -12 \\ \hline & 1 & -4 & 12 & 0 \end{array}$$

Now find the zeros of the depressed function: $x^2-4x+12=0$.

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(12)}}{2(1)}$$
$$=\frac{4\pm\sqrt{-32}}{2}=\frac{4\pm4i\sqrt{2}}{2}=2\pm2i\sqrt{2}$$

Solution set: $\left\{-1,\ 2\pm2i\sqrt{2}\right\}$

63. $3x^3-5x^2+16x+6=0$
The function has degree three, so there are three zeros. The possible rational zeros are $\left\{\pm1,\ \pm2,\ \pm3,\ \pm6,\ \pm\frac{1}{3},\ \pm\frac{2}{3}\right\}$. Using synthetic division, we find that one zero is $-1/3$:

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & -5 & 16 & 6 \\ & & -1 & 2 & -6 \\ \hline & 3 & -6 & 18 & 0 \end{array}$$

Now find the zeros of the depressed function: $3x^2-6x+18=0\Rightarrow3\left(x^2-2x+6\right)=0$.

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(6)}}{2(1)}$$
$$=\frac{2\pm\sqrt{-20}}{2}=\frac{2\pm2i\sqrt{5}}{2}=1\pm i\sqrt{5}$$

Solution set: $\left\{-1/3,\ 1\pm i\sqrt{5}\right\}$

64. $2x^3-9x^2+18x-7=0$
The function has degree three, so there are three zeros. The possible rational zeros are $\left\{\pm1,\ \pm7,\ \pm\frac{1}{2},\ \pm\frac{7}{2}\right\}$. Using synthetic division, we find that one zero is $1/2$:

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -9 & 18 & -7 \\ & & 1 & -4 & 7 \\ \hline & 2 & -8 & 14 & 0 \end{array}$$

Now find the zeros of the depressed function: $2x^2-8x+14=0\Rightarrow2\left(x^2-4x+7\right)=0$.

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(7)}}{2(1)}$$
$$=\frac{4\pm\sqrt{-12}}{2}=\frac{4\pm2i\sqrt{3}}{2}=2\pm i\sqrt{3}$$

Solution set: $\left\{\frac{1}{2},\ 2\pm i\sqrt{3}\right\}$

65. $x^4-x^3-x^2-x-2=0$
The function has degree four, so there are four zeros. The possible rational zeros are $\{\pm1,\ \pm2\}$. Using synthetic division, we find that one zero is 2:

$$\begin{array}{r|rrrrr} 2 & 1 & -1 & -1 & -1 & -2 \\ & & 2 & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & 1 & 0 \end{array}$$

Now find the zeros of the depressed function: $x^3+x^2+x+1=0$. We can use synthetic division or factor to find another zero:

$$x^3+x^2+x+1=0$$
$$x^2(x+1)+(x+1)=0$$
$$\left(x^2+1\right)(x+1)=0$$

$x^2+1=0\Rightarrow x^2=-1\Rightarrow x=\pm i$ or
$x+1=0\Rightarrow x=-1$
Solution set: $\{-1,\ 2,\ \pm i\}$

66. $x^4-x^3-13x^2+x+12=0$
The function has degree four, so there are four zeros. The possible rational zeros are $\{\pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm6,\ \pm12\}$. Using synthetic division, we find that one zero is 1:

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & -13 & 1 & 12 \\ & & 1 & 0 & -13 & -12 \\ \hline & 1 & 0 & -13 & -12 & 0 \end{array}$$

Now find the zeros of the depressed function: $x^3-13x-12=0$. Again using synthetic division, we find that one zero is –1:

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -13 & -12 \\ & & -1 & 1 & 12 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

Now find the zeros of the depressed function x^2-x-12.
$x^2-x-12=0\Rightarrow(x-4)(x+3)=0\Rightarrow x=4$ or $x=-3$. Solution set: $\{-3,-1,1,4\}$.

67. $2x^4 - x^3 - 2x^2 + 13x - 6 = 0$
The function has degree four, so there are four zeros. The possible rational zeros are $\left\{\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}\right\}$. Using synthetic division, we find that one zero is -2:

$$\begin{array}{r|rrrrr} -2 & 2 & -1 & -2 & 13 & -6 \\ & & -4 & 10 & -16 & 6 \\ \hline & 2 & -5 & 8 & -3 & 0 \end{array}$$

Now find the zeros of the depressed function: $2x^3 - 5x^2 + 8x - 3 = 0$. Again using synthetic division, we find that one zero is $1/2$:

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & -5 & 8 & -3 \\ & & 1 & -2 & 3 \\ \hline & 2 & -4 & 6 & 0 \end{array}$$

Now find the zeros of the depressed function: $2x^2 - 4x + 6 = 0 \Rightarrow 2\left(x^2 - 2x + 3\right) = 0$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2i\sqrt{2}}{2} = 1 \pm i\sqrt{2}$$

Solution set: $\left\{-2, \frac{1}{2}, 1 \pm i\sqrt{2}\right\}$.

68. $3x^4 - 14x^3 + 28x^2 - 10x - 7 = 0$
The function has degree four, so there are four zeros. The possible rational zeros are $\left\{\pm \frac{1}{3}, \pm 1, \pm \frac{7}{3}, \pm 7\right\}$. Using synthetic division, we find that one zero is 1.

$$\begin{array}{r|rrrrr} 1 & 3 & -14 & 28 & -10 & -7 \\ & & 3 & -11 & 17 & 7 \\ \hline & 3 & -11 & 17 & 7 & 0 \end{array}$$

Now find the zeros of the depressed function: $3x^3 - 11x^2 + 17x + 7 = 0$. Again using synthetic division, we find that one zero is $-1/3$:

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & -11 & 17 & 7 \\ & & -1 & 4 & -7 \\ \hline & 3 & -12 & 21 & 0 \end{array}$$

Now find the zeros of the depressed function: $3x^2 - 12x + 21 = 0 \Rightarrow 3\left(x^2 - 4x + 7\right) = 0$.

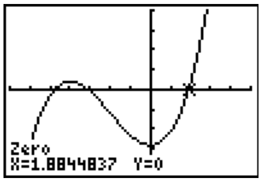
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{-12}}{2} = \frac{4 \pm 2i\sqrt{3}}{2} = 2 \pm i\sqrt{3}$$

Solution set: $\left\{-\frac{1}{3}, 1, 2 \pm i\sqrt{3}\right\}$.

69. The only possible rational roots are $\{\pm 1, \pm 2\}$. None of these satisfies the equation.

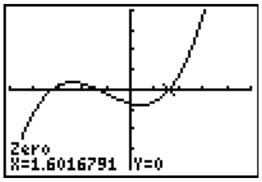
70. The only possible rational roots are $\left\{\pm \frac{1}{3}, \pm 1, \pm \frac{5}{3}, \pm 5\right\}$. None of these satisfies the equation.

71. $f(1) = 1^3 + 6(1)^2 - 28 = -21$;
$f(2) = 2^3 + 6(2)^2 - 28 = 4$. Because the sign changes, there is a real zero between 1 and 2. The zero is approximately 1.88.



Zero
X=1.8844837 Y=0

72. $f(1) = 1^3 + 3(1)^2 - 3(1) - 7 = -6$;
$f(2) = 2^3 + 3(2)^2 - 3(2) - 7 = 7$. Because the sign changes, there is a real zero between 1 and 2. The zero is approximately 1.60.



Zero
X=1.6016791 Y=0

73. $1 + \frac{1}{x} = 0 \Rightarrow x = -1$ is the x-intercept. There is no y-intercept. The vertical asymptote is the y-axis ($x = 0$). The horizontal asymptote is $y = 1$. Testing the intervals $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$, we find that the graph is above the x-axis on $(-\infty, -1) \cup (0, \infty)$ and below the x-axis on $(-1, 0)$.

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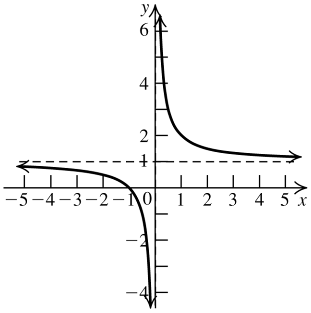
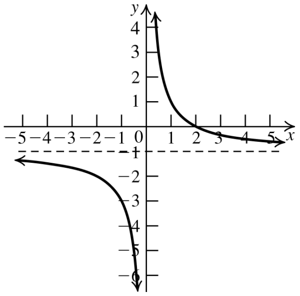
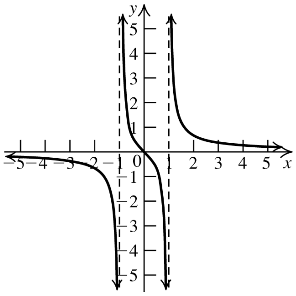


74. $\frac{2-x}{x}=0 \Rightarrow x=2$ is the *x*-intercept. There is no *y*-intercept. The vertical asymptote is the *y*-axis (*x* = 0). The horizontal asymptote is *y* = −1. Testing the intervals $(-\infty,0),(0,2),$ and $(2,\infty),$ we find that the graph is above the *x*-axis on $(0,2)$ and below the *x*-axis on $(-\infty,0)\cup(2,\infty).$



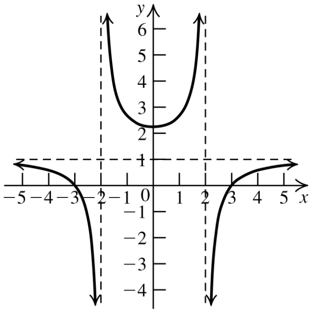
75. $\frac{x}{x^2-1}=0 \Rightarrow x=0$ is the *x*-intercept.

$\frac{0}{0^2-1}=0 \Rightarrow y=0$ is the *y*-intercept. The vertical asymptotes are *x* = 1 and *x* = −1. The horizontal asymptote is the *x*-axis. Testing the intervals $(-\infty,-1),(-1,0),\ (0,1),$ and $(1,\infty),$ we find that the graph is above the *x*-axis on $(-\infty,-1)\cup(0,\infty)$ and below the *x*-axis on $(-1,0)\cup(0,1).$



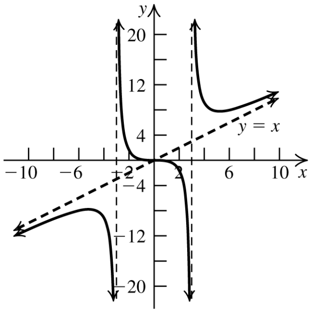
76. $\frac{x^2-9}{x^2-4}=0 \Rightarrow x=\pm3$ are the *x*-intercepts.

$\frac{0^2-9}{0^2-4}=\frac{9}{4} \Rightarrow y=\frac{9}{4}$ is the *y*-intercept. The vertical asymptotes are *x* = 2 and *x* = −2. The horizontal asymptote is *y* = 1. Testing the intervals $(-\infty,-3),\ (-3,-2),(-2,2),(2,3),$ and $(3,\infty),$ we find that the graph is above the *x*-axis on $(-\infty,-3)\cup(-2,2)\cup(3,\infty)$ and below the *x*-axis on $(-3,-2)\cup(2,3).$



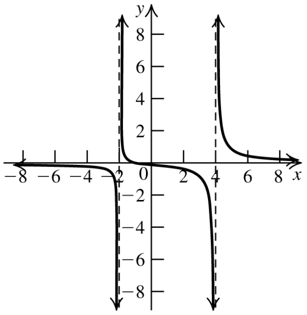
77. $\frac{x^3}{x^2-9}=0 \Rightarrow x=0$ is the *x*-intercept.

$\frac{0^3}{0^2-9}=0 \Rightarrow y=0$ is the *y*-intercept. The vertical asymptotes are *x* = 3 and *x* = −3. There is no horizontal asymptote. The oblique asymptote is *y* = *x*. Testing the intervals $(-\infty,-3),(-3,0),(0,3),$ and $(3,\infty),$ we find that the graph is above the *x*-axis on $(-3,0)\cup(3,\infty)$ and below the *x*-axis on $(-\infty,-3)\cup(0,3).$



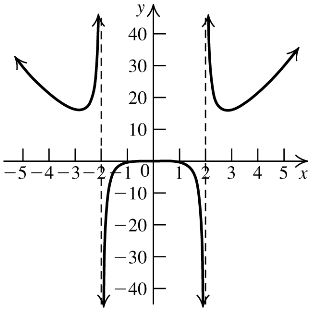
78. $\frac{x+1}{x^2-2x-8}=0 \Rightarrow x=-1$ is the x-intercept.

$\frac{0+1}{0^2-2(0)-8}=-\frac{1}{8} \Rightarrow y=-\frac{1}{8}$ is the y-intercept. The vertical asymptotes are $x = 4$ and $x = –2$. The horizontal asymptote is the x-axis. Testing the intervals $(-\infty,-2)$, $(-2,-1)$, $(-1,4)$, and $(4,\infty)$, we find that the graph is above the x-axis on $(-2,-1) \cup (4,\infty)$ and below the x-axis on $(-\infty,-2) \cup (-1,4)$.



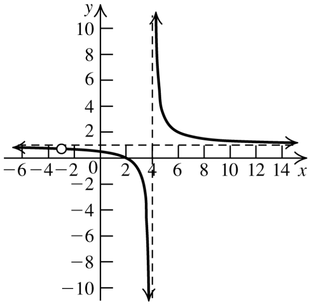
79. $\frac{x^4}{x^2-4}=0 \Rightarrow x=0$ is the x-intercept.

$\frac{0^4}{0^2-4}=0 \Rightarrow y=0$ is the y-intercept. The vertical asymptotes are $x = 2$ and $x = –2$. There is no horizontal asymptote. Testing the intervals $(-\infty,-2)$, $(-2,0)$, $(0,2)$, and $(2,\infty)$, we find that the graph is above the x-axis on $(-\infty,-2) \cup (2,\infty)$ and below the x-axis on $(-2,0) \cup (0,2)$.



80. $\frac{x^2+x-6}{x^2-x-12}=0 \Rightarrow x=2$ is the x-intercept.

$\frac{0^2+0-6}{0^2-0-12}=\frac{1}{2} \Rightarrow y=\frac{1}{2}$ is the y-intercept. The vertical asymptote is $x = 4$. The horizontal asymptote is $y = 1$. Testing the intervals $(-\infty,2)$, $(2,4)$, and $(4,\infty)$, we find that the graph is above the x-axis on $(-\infty,2) \cup (4,\infty)$ and below the x-axis on $(2,4)$.



Applying the Concepts

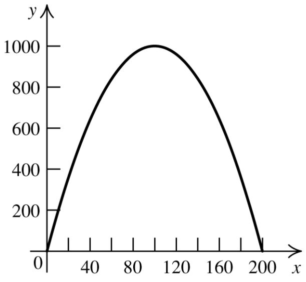
81. $y = kx; 12 = 4k \Rightarrow k = 3; y = 3(5) = 15$

82. $p=\frac{k}{q}; 4=\frac{k}{3} \Rightarrow 12=k; p=\frac{12}{4}=3$

83. $s = kt^2; 20 = 2^2k \Rightarrow 5 = k; s = 3^2(5) = 45$

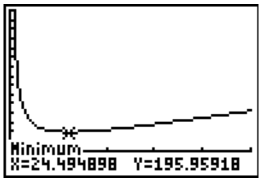
84. $y=\frac{k}{x^2}; 3=\frac{k}{8^2} \Rightarrow k=192; 12=\frac{192}{x^2} \Rightarrow x=\pm 4$

85. The maximum height occurs at the vertex, $\left(-\frac{20}{2(-1/10)} f\left(-\frac{20}{2(-1/10)}\right)\right)=(100,1000).$ The maximum height is 1000. To find where the missile hits the ground, solve $-\frac{1}{10}x^2+20x=0 \Rightarrow x=0$ or $x = 200$. The missile hits the ground at $x = 200$.



86. Let x = the length of one piece, and the area of the square formed by that piece is x^2. Then $20-x=$ the length of the other piece, $(20-x)^2=$ the area of the square formed by that piece. The total area is $x^2+(20-x)^2=2x^2-40x+400.$ The minimum is at the x-coordinate of the vertex, $-\frac{-40}{2(2)}=10.$ Each piece must be 10 cm long.

87. The area of each section is 400 square feet. Since the width is x, $y = \frac{400}{x}$. The total amount of fencing needed is $4x + \frac{2400}{x}$.
Using a graphing calculator, we find that this is a minimum at $x \approx 24.5$ feet. So
$y \approx \frac{400}{24.5} \approx 16.3$ feet.
The dimensions of each pen should be approximately 24.5 ft by 16.3 ft.

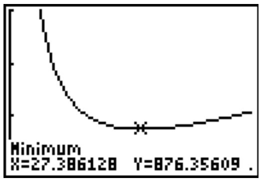


Minimum
X=24.494898 Y=195.95918

88. Since x = the width, $y = \frac{400}{x}$. Then

$6y = \frac{2400}{x} =$ the length of the heavy fencing.

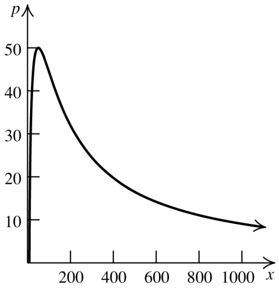
The farmer needs $2x + \frac{2400}{x}$ feet of heavy fencing, so it will cost
$5\left(2x + \frac{2400}{x}\right) = 10x + \frac{12,000}{x}$. The low fencing will cost $3(2x) = 6x$. The total cost of the fencing is $\frac{12,000}{x} + 16x$. Using a graphing calculator, we find this is a minimum when $x \approx 27.4$.



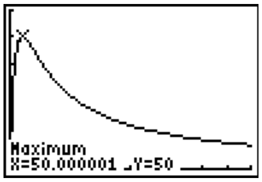
Minimum
X=27.386128 Y=876.35609

So $y \approx \frac{400}{27.4} \approx 14.6$. The cost will be minimized when the dimensions of each pen are 27.4 ft by 14.6 ft.

89. a.



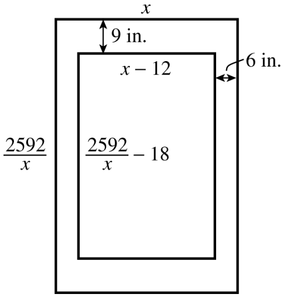
b.



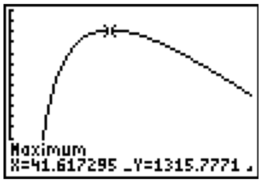
Maximum
X=50.000001 Y=50

The maximum occurs at $x = 50$.

90. The total area of the paper is 18 square feet = 18(144) = 2592 square inches. If the width of the paper is x, then the length is $2592/x$. The width of the printed area is $x - 12$, and the width of the printed area of $\frac{2592}{x} - 18$. The area of the printed region is
$(x-12)\left(\frac{2598}{x} - 18\right)$.



Using a graphing calculator, we find that this is a maximum at $x \approx 41.6$. So the dimensions of the paper are 41.6 in. by 62.4 in..



Maximum
X=41.617295 Y=1315.7771

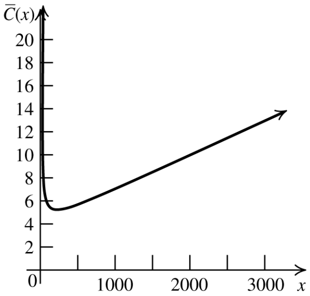
91. a. The revenue is $24x$. Profit = revenue – cost, so

$$P(x) = 24x - \left(150 + 3.9x + \frac{3}{1000}x^2\right)$$
$$= -\frac{3}{1000}x^2 + 20.1x - 150$$

b. The maximum occurs at the x-coordinate of the vertex: $-\frac{20.1}{2\left(-\frac{3}{1000}\right)} = 3350$

c. $\bar{C}(x) = \frac{C(x)}{x} = \frac{\frac{3}{1000}x^2 + 3.9x + 150}{x}$

$= \frac{3}{1000}x + 3.9 + \frac{150}{x}$



92. a. $19,340 \text{ feet} = \frac{19,340}{5280} = 3.66 \text{ miles}$

$p = \frac{69.1}{3.66 + 2.3} \approx 11.59 \text{ inches}$

b. $0 = \frac{69.1}{a + 2.3} \Rightarrow 0 = 69.1,$ which is impossible. So there is no altitude at which the pressure is 0.

93. $280 = 40k \Rightarrow k = 7; s = 7(35) = \$245$

94. $d = ks^2; 25 = 30^2 k \Rightarrow k = \frac{1}{36}$

$s = \frac{1}{36}(66^2) = 121 \text{ feet}$

95. $I = \frac{ki}{d^2},$ where I = illumination, i = intensity, and d = distance from the source. The illumination 6 inches from the source is $I_6 = \frac{300k}{6^2},$ while the illumination x inches from the source is $I_x = \frac{300k}{x^2}$. $I_6 = 2I_x \Rightarrow$

$\frac{300k}{36} = 2\left(\frac{300k}{x^2}\right) \Rightarrow x^2 = 72 \Rightarrow x = 6\sqrt{2}$ in.

96. $V = kT; 1.2 = 295k \Rightarrow k = \frac{1.2}{295}$

$V = \frac{1.2}{295}(310) \approx 1.26$ cubic feet.

97. $I = \frac{k}{R}; 30 = \frac{k}{300} \Rightarrow k = 9000$

a. $I = \frac{9000}{250} = 36$ amp

b. $60 = \frac{9000}{R} \Rightarrow R = 150$ ohms

98. $L = \frac{k\sqrt{r}}{l^2}; 20 = \frac{k\sqrt{4}}{12^2} \Rightarrow 1440 = k$

$L = \frac{1440\sqrt{3}}{10^2} \approx 24.94$ tons

99. $R = ki(p - i)$

a. $255 = k(0.15)(20,000)(0.85)(20,000)$

$= \frac{1}{200,000}$

b. $R = \frac{1}{200,000}(10,000)(10,000) = 500$ people per day.

c. $95 = \frac{1}{200,000}x(20,000 - x) \Rightarrow$

$x^2 - 20,000x + 19,000,000 = 0 \Rightarrow$
$(x - 1000)(x - 19,000) = 0 \Rightarrow x = 1000$ or
$x = 19,000$

100. $F = (kq_1q_2)/d^2$. If the distance is quadrupled, then the force is

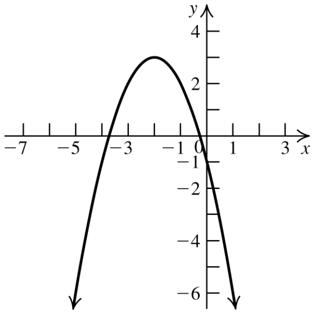
$F_{\text{new}} = \frac{kq_1q_2}{(4d)^2} = \frac{kq_1q_2}{16d^2}$. So the original force is divided by 16: $\frac{96}{16} = 6$ units.

Chapter 3 Practice Test A

1. $x^2 - 6x + 2 = 0 \Rightarrow x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(2)}}{2(1)} \Rightarrow$

$x = \frac{6 \pm \sqrt{28}}{2} = 3 \pm \sqrt{7}$ are the x-intercepts

2.



3. The vertex is at $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

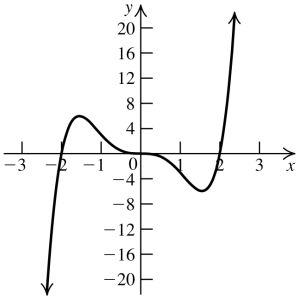
$$= \left(-\frac{14}{2(-7)}, f\left(-\frac{14}{2(-7)}\right)\right) = (1,10).$$

4. The denominator is 0 when $x = -4$ or $x = 1$. The domain is $(-\infty, -4) \cup (-4,1) \cup (1,\infty)$.

5. Using either long division or synthetic division, we find that the quotient is $x^2 - 4x + 3$ and the remainder is 0.

$$\begin{array}{r|rrrr} \underline{-2} & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

6.



7. The function has degree three, so there are three zeros. Since 2 is a zero, use synthetic division to find the depressed function:

$$\begin{array}{r|rrrr} \underline{2} & 2 & -2 & -8 & 8 \\ & & 4 & 4 & -8 \\ \hline & 2 & 2 & -4 & 0 \end{array}$$

Now find the zeros of $2x^2 + 2x - 4$:

$2x^2 + 2x - 4 = 2(x^2 + x - 2) = 2(x+2)(x-1) \Rightarrow$ the zeros are $x = -2$ and $x = 1$. The zeros of the original function are –2, 1, 2.

8.

$$\begin{array}{r@{}l} & \;-3x^2 + 5x + 1 \\ 2x+3 & \overline{\big)-6x^3 + \;\; x^2 + 17x + 3} \\ & \underline{-6x^3 - 9x^2} \\ & \qquad\quad 10x^2 + 17x \\ & \qquad\quad \underline{10x^2 + 15x} \\ & \qquad\qquad\qquad 2x + 3 \\ & \qquad\qquad\qquad \underline{2x + 3} \\ & \qquad\qquad\qquad\qquad 0 \end{array}$$

9. Using synthetic division to find the remainder, we have $P(-2) = -53$.

$$\begin{array}{r|rrrrr} \underline{-2} & 1 & 5 & -7 & 9 & 17 \\ & & -2 & -6 & 26 & -70 \\ \hline & 1 & 3 & -13 & 35 & -53 \end{array}$$

10. The function has degree three, so there are three zeros. There are two sign changes in $f(x)$, so there are either 2 or 0 positive zeros. There is one sign change in $f(-x)$, so there is one negative zero. The possible rational zeros are $\{\pm1, \pm2, \pm4, \pm5, \pm10, \pm20\}$. Using synthetic division, we find that –2 is a zero:

$$\begin{array}{r|rrrr} \underline{-2} & 1 & -5 & -4 & 20 \\ & & -2 & 14 & -20 \\ & 1 & -7 & 10 & 0 \end{array}$$

Now find the zeros of the depressed function $x^2 - 7x + 10 : x^2 - 7x + 10 = (x-5)(x-2) \Rightarrow$ the zeros are $x = 2$ and $x = 5$. The zeros of the original function are –2, 2, 5.

11. The function has degree four, so there are four zeros. Factoring, we have $x^4 + x^3 - 15x^2$ $= x^2(x^2 + x - 15) \Rightarrow 0$ is a zero of multiplicity 2. Now find the zeros of $x^2 + x - 15$:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-15)}}{2(1)} = \frac{-1 \pm \sqrt{61}}{2}.$$

The zeros of the original function are 0 and $\frac{-1 \pm \sqrt{61}}{2}$.

12. The factors of the constant term, 9, are $\{\pm1, \pm3, \pm9\}$, while the factors of the leading coefficient are $\{\pm1, \pm2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm3, \pm\frac{9}{2}, \pm9\right\}$.

13. $f(x) \to -\infty$ as $x \to -\infty$; $f(x) \to \infty$ as $x \to \infty$.

14. $f(x) = (x^2 - 4)(x+2)^2$

$= (x-2)(x+2)(x+2)^2 \Rightarrow$

the zeros are –2 (multiplicity 3) and 2 (multiplicity 1).

15. There are two sign changes in $f(x)$, so there are either two or zero positive zeros. There is one sign change in $f(-x)$, so there is one negative zero.

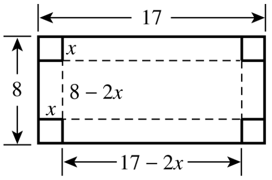
16. The zeros of the denominator are $x = 5$ and $x = -4$, so those are the vertical asymptotes. The degree of the numerator equals the degree of the denominator, so the horizontal asymptote is $y = 2$.

17. $y = \frac{kx}{t^2}$

18. $6=\frac{8k}{2^2}\Rightarrow k=3;\ y=\frac{3(12)}{3^2}=4$

19. The minimum occurs at the x-coordinate of the vertex: $-\frac{-30}{2(1)}=15$ thousand units.

20. $V=x(17-2x)(8-2x)$



Chapter 3 Practice Test B

1. $x^2+5x+3=0\Rightarrow x=\frac{-5\pm\sqrt{5^2-4(1)(3)}}{2(1)}$

$=\frac{-5\pm\sqrt{13}}{2}$. The answer is B.

2. The graph of $f(x)=4-(x-2)^2$ is the graph of $f(x)=x^2$ shifted two units right, reflected across the x-axis, and then shifted 4 units up. The answer is D.

3. $\left(-\frac{b}{2a},f\left(-\frac{b}{2a}\right)\right)=\left(-\frac{12}{2(6)},f\left(-\frac{12}{2(6)}\right)\right)$

$=(-1,-11)$. The answer is A.

4. The denominator is 0 when $x=-3$ or $x=2$. The answer is B.

5.
$$\begin{array}{r|rrrr} -3 & 1 & 0 & -8 & 6 \\ & & -3 & 9 & -3 \\ \hline & 1 & -3 & 1 & 3 \end{array}$$
The answer is D.

6. $P(x)=x^4+2x^3=x^3(x+2)$. So the zeros are 0 (multiplicity 3) and –2 (multiplicity 1). The only graph with those zeros is C.

7.
$$\begin{array}{r|rrrr} 3 & 3 & -26 & 61 & -30 \\ & & 9 & -51 & 30 \\ \hline & 3 & -17 & 10 & 0 \end{array}$$
The zeros of the depressed function $3x^2-17x+10$ are $x=\frac{2}{3}$ and $x=5$.

The answer is C.

8.
$$\begin{array}{rl}
 & \phantom{-10x^3+{}}-5x^2+3x-4 \\
2x-3 & \overline{\smash{)}-10x^3+21x^2-17x+12} \\
 & \underline{-10x^3+15x^2} \\
 & \phantom{-10x^3+{}}6x^2-17x \\
 & \phantom{-10x^3+{}}\underline{6x^2-\ \ 9x} \\
 & \phantom{-10x^3+6x^2{}}-8x+12 \\
 & \phantom{-10x^3+6x^2{}}\underline{-8x+12} \\
 & 0
\end{array}$$
The answer is B.

9.
$$\begin{array}{r|rrrrr} -3 & 1 & 4 & 7 & 10 & 15 \\ & & -3 & -3 & -12 & 6 \\ \hline & 1 & 1 & 4 & -2 & 21 \end{array}$$
$P(-3)=21$. The answer is C.

10. The polynomial has degree three, so there are three zeros. The possible rational zeros are $\{\pm1,\pm2,\pm3,\pm4,\pm6,\pm12\}$. Using synthetic division, we find that one zero is –3:
$$\begin{array}{r|rrrr} -3 & -1 & 1 & 8 & -12 \\ & & 3 & -12 & 12 \\ \hline & -1 & 4 & -4 & 0 \end{array}$$
The zeros of the depressed function $-x^2+4x-4$ are $x=2$ (multiplicity 2). The answer is A.

11. The polynomial has degree three, so there are three zeros. Factoring, we have
$x^3+x^2-30x=x(x^2+x-30)=x(x+6)(x-5)$
The answer is A.

12. The factors of the constant term, 60, are $\{\pm1,\pm2,\pm3,\pm4,\pm5,\pm6,\pm10,\pm12,\pm15,\pm20,$ $\pm30,\pm60\}$. Since the leading coefficient is 1, these are also the possible rational zeros. The answer is D.

13. The answer is C.

14. $(x^2-1)(x+1)^2=(x-1)(x+1)(x+1)^2$.
The answer is B.

15. There are three sign changes in $P(x)$, so there are 3 or 1 positive zeros. There are two sign changes in $P(-x)$, so there are 2 or 0 negative zeros. The answer is C.

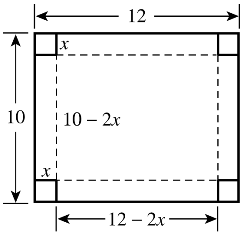
16. The zeros of the denominator are $x=3$ and $x=-4$, so those are the vertical asymptotes. The degree of the numerator equals the degree of the denominator, so the horizontal asymptote is $y=1$. The answer is C.

17. The answer is D.

18. $27 = \frac{3^2 k}{1^3} \Rightarrow k = 3; S = \frac{3(6^2)}{3^3} = 4$.
The answer is B.

19. The minimum occurs at the x-coordinate of the vertex: $-\frac{-24}{2(1)} = 12$. The answer is B.

20. $V = x(10-2x)(12-2x)$. The answer is D.

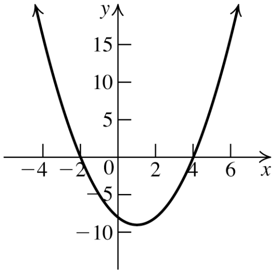


Cumulative Review Exercises
Chapters P–3

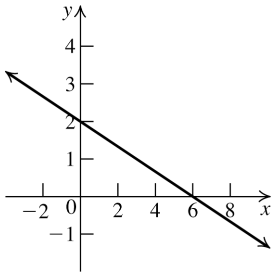
1. $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)}$$
$$= \sqrt{(-1-2)^2 + (3-5)^2} = \sqrt{13}$$

2. $$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{2+(-8)}{2}, \frac{-5+(-3)}{2}\right) = (-3, -4)$$

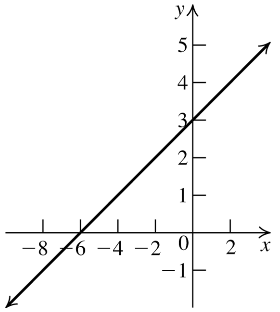
3. $0 = x^2 - 2x - 8 \Rightarrow 0 = (x-4)(x+2) \Rightarrow$
$x - 4 = 0 \Rightarrow x = 4$ or $x + 2 = 0 \Rightarrow x = -2$
$f(0) = 0^2 - 2(0) - 8 = -8$
The x-intercepts are (–2, 0) and (4, 0), and the y-intercept is (0, –8).



4. Write the equation in slope-intercept form:
$x + 3y - 6 = 0 \Rightarrow 3y = -x + 6 \Rightarrow y = -\frac{1}{3}x + 2$
The slope is $-\frac{1}{3}$, and the y-intercept is (0, 2).
$x + 3(0) - 6 = 0 \Rightarrow x - 6 = 0 \Rightarrow x = 6,$ so the x-intercept is (6, 0).



5. Write the equation in slope-intercept form:
$x = 2y - 6 \Rightarrow x + 6 = 2y \Rightarrow \frac{1}{2}x + 3 = y$
The slope is $\frac{1}{2}$, and the y-intercept is (0, 3).
The x-intercept is $x = 2(0) - 6 = -6$.



6. $(x-2)^2 + (y+3)^2 = 16$

7. $x^2 + y^2 + 2x - 4y - 4 = 0 \Rightarrow$
$(x^2 + 2x) + (y^2 - 4y) = 4 \Rightarrow$
$(x^2 + 2x + 1) + (y^2 - 4y + 4) = 4 + 1 + 4 \Rightarrow$
$(x+1)^2 + (y-2)^2 = 9$. The center is (–1, 2).
The radius is 3.

8. $y + 2 = 3(x-1) \Rightarrow y = 3x - 5$

9. $2x + 3y = 5 \Rightarrow y = -\frac{2}{3}x + \frac{5}{3} \Rightarrow$ the slope is $-\frac{2}{3}$.
$y - 3 = -\frac{2}{3}(x-1) \Rightarrow y = -\frac{2}{3}x + \frac{11}{3}$.

10. $2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$.
The domain is $\left(-\infty, -\frac{3}{2}\right) \cup \left(-\frac{3}{2}, \infty\right)$.

11. $\sqrt{4-2x} = 0 \Rightarrow x = 2.$ The domain is $(-\infty, 2)$.

12.
$$\begin{aligned} f(-2) &= (-2)^2 - 2(-2) + 3 = 11; \\ f(3) &= 3^2 - 2(3) + 3 = 6; \\ f(x+h) &= (x+h)^2 - 2(x+h) + 3 \\ &= x^2 + 2xh + h^2 - 2x - 2h + 3 \\ &= x^2 + 2(h-1)x + h^2 - 2h + 3 \end{aligned}$$

$$\begin{aligned} &\frac{f(x+h) - f(x)}{h} \\ &= \frac{(x^2 + 2(h-1)x + h^2 - 2h + 3) - (x^2 - 2x + 3)}{h} \\ &= \frac{2(h-1)x + 2x + h^2 - 2h}{h} \\ &= \frac{2hx - 2x + 2x + h^2 - 2h}{h} = \frac{h^2 + 2hx - 2h}{h} \\ &= h + 2x - 2 \end{aligned}$$

13. a. $f\left(g(x)\right) = \sqrt{x^2 + 1}$

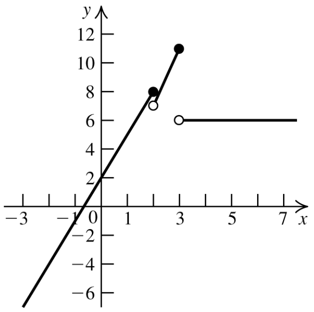
b. $g\left(f(x)\right) = \left(\sqrt{x}\right)^2 + 1 = x + 1$

c. $f\left(f(x)\right) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$

d. $g\left(g(x)\right) = \left(x^2 + 1\right)^2 + 1 = x^4 + 2x^2 + 2$

14. a. $f(1) = 3(1) + 2 = 5; f(3) = 4(3) - 1 = 11;$
$f(4) = 6$

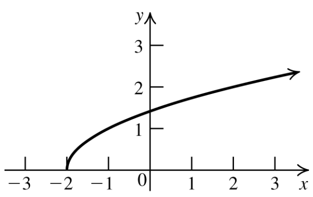
b.



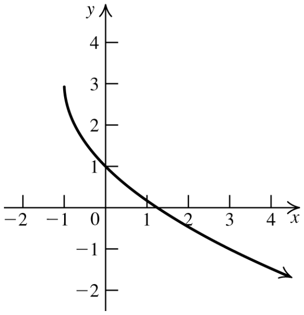
15. $y = 2x - 3$. Interchange *x* and *y*, and then solve for *y*.

$$x = 2y - 3 \Rightarrow y = \frac{x+3}{2} = \frac{1}{2}x + \frac{3}{2} = f^{-1}(x).$$

16. a. Shift the graph of $y = \sqrt{x}$ two units left.

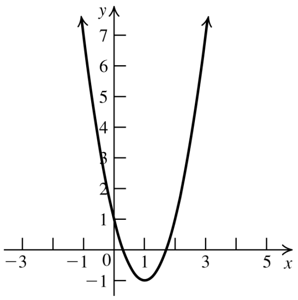


b. Shift the graph of $y = \sqrt{x}$ one unit left, stretch vertically by a factor of two, reflect about the *x*-axis, and then shift up three units.

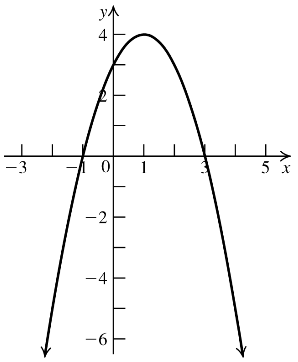


17. The factors of the constant term, –6, are $\{\pm1, \pm2, \pm3, \pm6\}$, and the factors of the leading coefficient, 2, are $\{\pm1, \pm2\}$. The possible rational zeros are $\left\{\pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \pm2, \pm3, \pm6\right\}$.

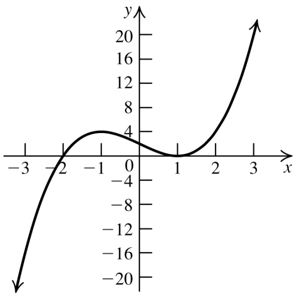
18.



19.



20.



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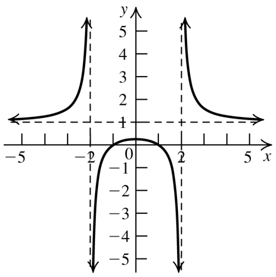

22. Since one zero is 1 + *i*, another zero is 1 − *i*. So $(x-(1-i))(x-(1+i)) = x^2-2x+2$ is a factor of $f(x)$. Now divide to find the other factor:

$$\begin{array}{r}
x^2 - x - 2 \\
x^2-2x+2\overline{\big)\,x^4-3x^3+2x^2+2x-4} \\
\underline{x^4-2x^3+2x^2} \\
-x^3+0x^2+2x \\
\underline{-x^3+2x^2-2x} \\
-2x^2+4x-4 \\
\underline{-2x^2+4x-4} \\
0
\end{array}$$

$$\begin{aligned}
f(x) &= x^4-3x^3+2x^2+2x-4 \\
&= (x-(1-i))(x-(1+i))(x^2-x-2) \\
&= (x-(1-i))(x-(1+i))(x-2)(x+1) \Rightarrow
\end{aligned}$$

the zeros of $f(x)$ are $-1, 2, 1-i, 1+i$.

23. $y = k\sqrt{x}; 6 = k\sqrt{4} \Rightarrow k = 3; y = 3\sqrt{9} = 9$

24. Profit = revenue – cost

$$\begin{aligned}
&150x-(0.02x^2+100x+3000) \\
&\qquad = -0.02x^2+50x-3000.
\end{aligned}$$

The maximum occurs at the vertex:

$$\left(-\frac{50}{2(-0.02)}, f\left(-\frac{50}{2(-0.02)}\right)\right) = (1250, \$28,250).$$

25.

$$\begin{array}{r|rrrr}
\underline{10} & 0.02 & 48.8 & -2990 & 25,000 \\
 & & 0.2 & 490 & -25,000 \\
\hline
 & 0.02 & 49 & -2500 & 0
\end{array}$$

Now solve the depressed equation $0.02x^2+49x-2500=0$.

$$\begin{aligned}
x &= \frac{-49 \pm \sqrt{49^2-4(0.02)(-2500)}}{2(0.02)} \\
&= \frac{-49 \pm \sqrt{2601}}{0.04} = \frac{-49 \pm 51}{0.04} \Rightarrow x = 50 \text{ or}
\end{aligned}$$

$x = -2500$. There cannot be a negative amount of units sold, so another break-even point is 50.