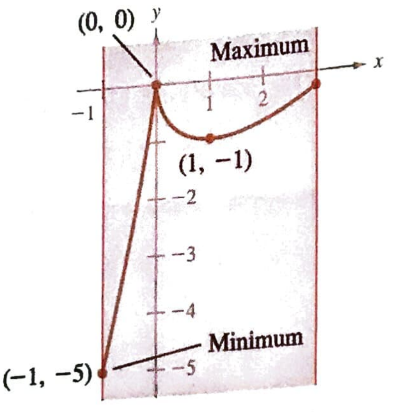


$f(x) = 2x - 3x^{2/3}$

FIGURE 4.6

SOLUTION

Differentiating produces

$$f'(x) = 2 - \frac{2}{x^{1/3}} = 2\left(\frac{x^{1/3} - 1}{x^{1/3}}\right).$$

This gives us the following critical numbers.

$x = 1 \qquad f'(1) = 0$

$x = 0 \qquad f'$ is undefined

Finally, by evaluating f at these two points and at the endpoints of the interval, we conclude that the minimum is $f(-1) = -5$ and the maximum is $f(0) = 0$, as indicated in Table 4.2. The graph of f is shown in Figure 4.6.

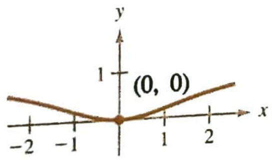
TABLE 4.2

Left endpoint	*Critical number*	*Critical number*	*Right endpoint*
$f(-1) = -5$ Minimum	$f(0) = 0$ Maximum	$f(1) = -1$	$f(3) = 6 - 3\sqrt[3]{9} \approx -0.24$

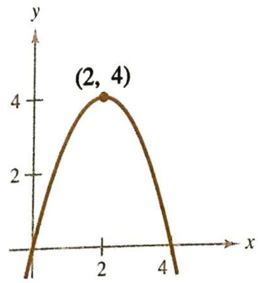
EXERCISES for Section 4.1

In Exercises 1–6, find the value of the derivative (if it exists) at the indicated extrema.

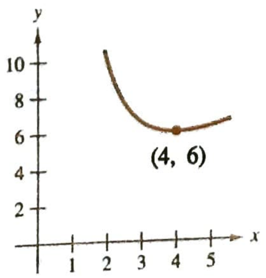
1. $f(x) = \dfrac{x^2}{x^2 + 4}$



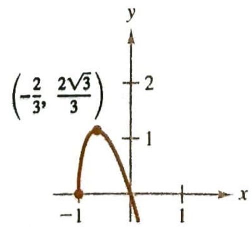
2. $f(x) = -x^2 + 4x$



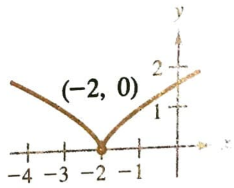
3. $f(x) = x + \dfrac{32}{x^2}$



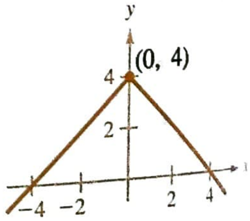
4. $f(x) = -3x\sqrt{x + 1}$



5. $f(x) = (x + 2)^{2/3}$



6. $f(x) = 4 - |x|$



In Exercises 7–18, locate the absolute extrema of the given function on the indicated interval.

Function	*Interval*
7. $f(x) = 2(3 - x)$	$[-1, 2]$
8. $f(x) = \dfrac{2x + 5}{3}$	$[0, 5]$
9. $f(x) = -x^2 + 3x$	$[0, 3]$
10. $f(x) = x^2 + 2x - 4$	$[-1, 1]$
11. $f(x) = x^3 - 3x^2$	$[-1, 3]$
12. $f(x) = x^3 - 12x$	$[0, 4]$
13. $f(x) = 3x^{2/3} - 2x$	$[-1, 1]$
14. $g(x) = \sqrt[3]{x}$	$[-1, 1]$
15. $h(t) = 4 - \lvert t - 4\rvert$	$[1, 6]$

16. $g(t) = \dfrac{t^2}{t^2 + 3}$ $[-1, 1]$

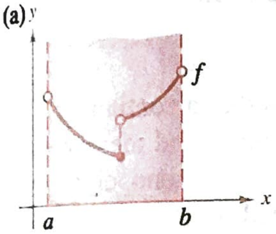
17. $h(s) = \dfrac{1}{s - 2}$ $[0, 1]$

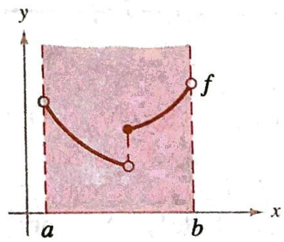
18. $h(t) = \dfrac{t}{t - 2}$ $[3, 5]$

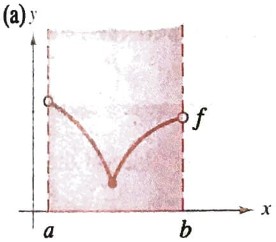
19. Explain why the function $f(x) = 1/x^2$ has a maximum on $[1, 2]$ but not on $(0, 2]$.

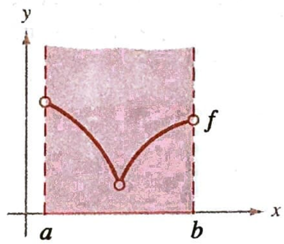
20. Explain why the function $y = 1/(x + 1)$ has a minimum on $[0, 2]$, but not on $[-2, 0]$.

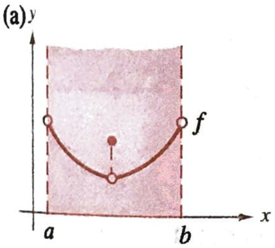
In Exercises 21–24, determine from the graph whether f possesses a minimum in the interval (a, b).

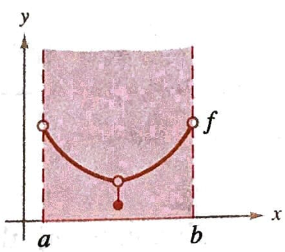
21. (a)


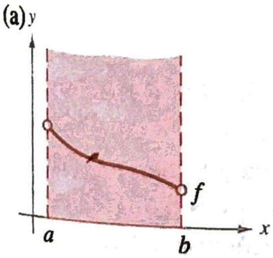
(b)


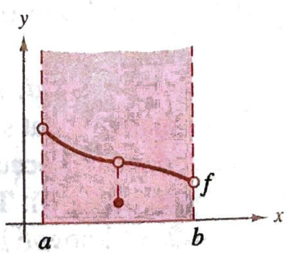
22. (a)


(b)


23. (a)


(b)


24. (a)


(b)


In Exercises 25 and 26, locate the absolute extrema of the function (if any exist) over the indicated interval.

25. $f(x) = 2x - 3$
(a) $[0, 2]$
(b) $[0, 2)$
(c) $(0, 2]$
(d) $(0, 2)$

26. $f(x) = 5 - x$
(a) $[1, 4]$
(b) $[1, 4)$
(c) $(1, 4]$
(d) $(1, 4)$

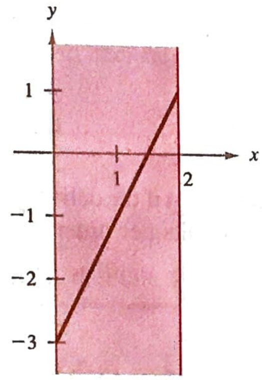


FIGURE FOR 25

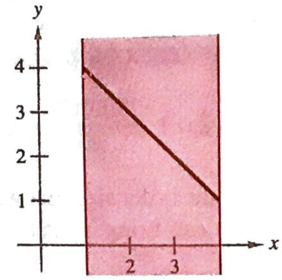


FIGURE FOR 26

In Exercises 27–30, find the maximum value of $|f''(x)|$ on the indicated interval. (This value is used in the error estimate for the Trapezoidal Rule, as discussed in Section 5.6.)

Function	*Interval*
27. $f(x) = \dfrac{1}{x^2 + 1}$	$[0, 3]$
28. $f(x) = \dfrac{1}{x^2 + 1}$	$\left[\dfrac{1}{2}, 3\right]$
29. $f(x) = \sqrt{1 + x^3}$	$[0, 2]$
30. $f(x) = x^3(3x^2 - 10)$	$[0, 1]$

In Exercises 31–34, find the maximum value of $|f^{(4)}(x)|$ on the indicated interval. (This value is used in the error estimate for Simpson's Rule, as discussed in Section 5.6.)

Function	*Interval*
31. $f(x) = 15x^4 - \left(\dfrac{2x - 1}{2}\right)^6$	$[0, 1]$
32. $f(x) = x^5 - 5x^4 + 20x^3 + 600$	$\left[0, \dfrac{3}{2}\right]$
33. $f(x) = (x + 1)^{2/3}$	$[0, 2]$
34. $f(x) = \dfrac{1}{x^2 + 1}$	$[-1, 1]$

35. The formula for the power output P of a battery is given by

$$P = VI - RI^2$$

where V is the electromotive force in volts, R is the resistance, and I is the current. Find the current (measured in amperes) that corresponds to a maximum value of P in a battery for which $V = 12$ volts and $R = 0.5$ ohms. (Assume that a 15-amp fuse bounds the output in the interval $0 \le I \le 15$.)

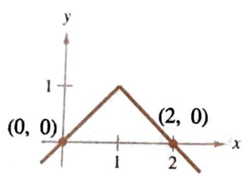
REMARK A useful alternative form of the Mean Value Theorem is as follows: If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that $f(b) = f(a) + (b - a)f'(c)$.

When working the exercises for this section, keep in mind that polynomial functions and rational functions are differentiable at all points in their domains.

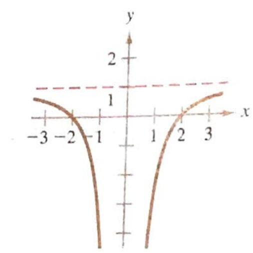
EXERCISES for Section 4.2

In Exercises 1 and 2, state why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b) = 0$.

1. $f(x) = 1 - |x - 1|$



2. $f(x) = \dfrac{x^2 - 4}{x^2}$



In Exercises 3–12, determine whether Rolle's Theorem can be applied to f on the indicated interval. If Rolle's Theorem can be applied, find all values of c in the interval such that $f'(c) = 0$.

Function	Interval
3. $f(x) = x^2 - 2x$	$[0, 2]$
4. $f(x) = x^2 - 3x + 2$	$[1, 2]$
5. $f(x) = (x - 1)(x - 2)(x - 3)$	$[1, 3]$
6. $f(x) = (x - 3)(x + 1)^2$	$[-1, 3]$
7. $f(x) = \lvert x \rvert - 1$	$[-1, 1]$
8. $f(x) = 3 - \lvert x - 3 \rvert$	$[0, 6]$
9. $f(x) = x^{2/3} - 1$	$[-8, 8]$
10. $f(x) = x - x^{1/3}$	$[0, 1]$
11. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$	$[-1, 3]$
12. $f(x) = \dfrac{x^2 - 1}{x}$	$[-1, 1]$

In Exercises 13–20, apply the Mean Value Theorem to f on the indicated interval. In each case, find all values of c in the interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Function	Interval
13. $f(x) = x^2$	$[-2, 1]$
14. $f(x) = x(x^2 - x - 2)$	$[-1, 1]$
15. $f(x) = x^{2/3}$	$[0, 1]$
16. $f(x) = \dfrac{x + 1}{x}$	$\left[\dfrac{1}{2}, 2\right]$
17. $f(x) = \dfrac{x}{x + 1}$	$\left[-\dfrac{1}{2}, 2\right]$
18. $f(x) = \sqrt{x - 2}$	$[2, 6]$
19. $f(x) = x^3$	$[0, 1]$
20. $f(x) = x^3 - 2x$	$[0, 2]$

21. The height of a ball t seconds after it is thrown is given by

$$f(t) = -16t^2 + 48t + 32.$$

(a) Verify that $f(1) = f(2)$.
(b) According to Rolle's Theorem, what must be the velocity at some time in the interval $[1, 2]$?

22. The ordering and transportation cost C of components used in a manufacturing process is approximated by

$$C(x) = 10\left(\frac{1}{x} + \frac{x}{x + 3}\right)$$

where C is measured in thousands of dollars and x is the order size in hundreds.
(a) Verify that $C(3) = C(6)$.
(b) According to Rolle's Theorem, the rate of change of cost must be zero for some order size in the interval $[3, 6]$. Find that order size.

23. The height of an object t seconds after it was dropped from a height of 500 feet is given by

$$s(t) = -16t^2 + 500.$$

(a) Find the average velocity of the object during the first 3 seconds.
(b) Use the Mean Value Theorem to verify that at some time during the first three seconds of fall the instantaneous velocity equals the average velocity. Find that time.

24. A company introduces a new product for which the number of units sold S is given by

$$S(t) = 200\left(5 - \frac{9}{2+t}\right)$$

where t is the time in months.
(a) Find the average rate of change of $S(t)$ during the first year.
(b) During what month does $S'(t)$ equal its average rate of change during the first year?

25. Given the function

$$f(x) = \frac{1}{x-4}$$

show that for the interval $(2, 6)$ there exists no real number c such that

$$f'(c) = \frac{f(6) - f(2)}{6 - 2}.$$

State whether this contradicts the Mean Value Theorem and give the reason for your answer.

26. Prove the Corollary to Rolle's Theorem.

27. If $a > 0$ and n is any integer, prove that the polynomial function

$$p(x) = x^{2n+1} + ax + b$$

cannot have two real roots.

28. Let p be a *nonconstant* polynomial function.
(a) Prove that between any two consecutive zeros of p', there is at most one zero of p.
(b) If p has three distinct zeros in the interval $[a, b]$, prove that $p''(c) = 0$ for some real number c in (a, b).

29. Prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on the interval.

30. Let $p(x) = Ax^2 + Bx + C$. Prove that for any interval $[a, b]$, the value c guaranteed by the Mean Value Theorem is the midpoint of the interval.

31. Prove that if two functions f and g have the same derivatives on an interval, then they must differ only by a constant on the interval. [Hint: Let $h(x) = f(x) - g(x)$ and use the result of Exercise 29.]

32. Prove that if f is differentiable on $(-\infty, \infty)$ and $f'(x) < 1$ for all real numbers, then f has at most one fixed point. A **fixed point** of a function f is a real number c such that $f(c) = c$.

33. Use a computer or graphics calculator to sketch the graph of $f(x) = \sqrt{x}$ over the interval $[1, 9]$.
(a) Find an equation of the secant line to the graph of f passing through the points $(1, f(1))$ and $(9, f(9))$. Sketch the graph of the secant line on the same axes as the graph of f.
(b) Find the value of c in the interval $(1, 9)$ such that

$$f'(c) = \frac{f(9) - f(1)}{9 - 1}.$$

Find the equation of the tangent line to the graph of f at the point $(c, f(c))$ and sketch its graph on the same axes as the graph of f. Note that the secant line and tangent line are parallel.

4.3 Increasing and Decreasing Functions and the First Derivative Test

Increasing and decreasing functions ▪ The First Derivative Test ▪ Strictly monotonic functions

We now know that the derivative is useful in *locating* the relative extrema of a function. In this section we will show that the derivative also can be used to *classify* relative extrema as either relative minima or relative maxima. We begin by defining what is meant when we say a function increases (or decreases) on an interval.

DEFINITION OF INCREASING AND DECREASING FUNCTIONS

A function f is said to be **increasing** on an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \quad \text{implies} \quad f(x_1) < f(x_2).$$

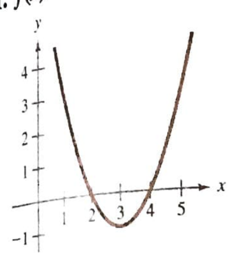
A function f is said to be **decreasing** on an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \quad \text{implies} \quad f(x_1) > f(x_2).$$

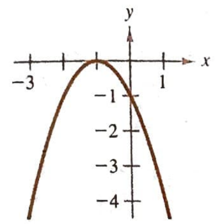
EXERCISES for Section 4.3

In Exercises 1–6, identify the open intervals on which the function is increasing or decreasing.

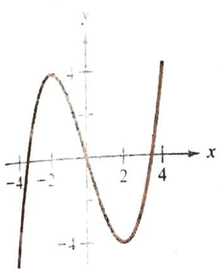
1. $f(x) = x^2 - 6x + 8$



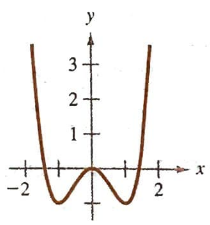
2. $y = -(x + 1)^2$



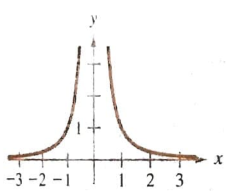
3. $y = \frac{x^3}{4} - 3x$



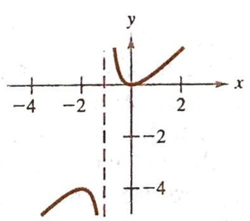
4. $f(x) = x^4 - 2x^2$



5. $f(x) = \frac{1}{x^2}$



6. $y = \frac{x^2}{x + 1}$



In Exercises 7–18, find the critical numbers of f (if any), find the open intervals on which f is increasing or decreasing, and locate all relative extrema.

7. $f(x) = -2x^2 + 4x + 3$
8. $f(x) = x^2 + 8x + 10$
9. $f(x) = x^2 - 6x$
10. $f(x) = (x - 1)^2(x + 2)$
11. $f(x) = 2x^3 + 3x^2 - 12x$
12. $f(x) = (x - 3)^3$
13. $f(x) = x^{1/3} + 1$
14. $f(x) = x^{2/3}(x - 5)$
15. $f(x) = \frac{x^2}{x^2 - 9}$
16. $f(x) = \frac{x + 3}{x^2}$
17. $f(x) = \frac{x^5 - 5x}{5}$
18. $f(x) = x^4 - 32x + 4$

In Exercises 19–26, find the critical numbers of f (if any), find the open intervals on which the algebraic function is increasing or decreasing, and locate all relative extrema. Use a graphing utility to confirm your results.

19. $f(x) = x^3 - 6x^2 + 15$
20. $f(x) = x^4 - 2x^3$
21. $f(x) = (x - 1)^{2/3}$
22. $f(x) = (x - 1)^{1/3}$
23. $f(x) = x + \frac{1}{x}$
24. $f(x) = \frac{x}{x + 1}$
25. $f(x) = \frac{x^2 - 2x + 1}{x + 1}$
26. $f(x) = \frac{x^2 - 3x - 4}{x - 2}$

In Exercises 27 and 28, determine whether the given function is strictly monotonic on the indicated interval.

27. $f(x) = x^2$
    (a) $(-\infty, \infty)$ (b) $(-\infty, 0)$ (c) $(0, \infty)$
28. $f(x) = x^3 - x$
    (a) $(-1, 0)$ (b) $\left(-1, -\frac{1}{2}\right)$ (c) $(-1, 1)$

29. The height (in feet) of a ball at time t (in seconds) is given by the position function

$$s(t) = 96t - 16t^2.$$

Find the open interval on which the ball is moving up and the open interval on which it is moving down. What is the maximum height of the ball?

30. Repeat Exercise 29 using the position function

$$s(t) = -16t^2 + 64t.$$

31. Coughing forces the trachea (windpipe) to contract, which affects the velocity v of the air passing through the trachea. Suppose the velocity of the air during coughing is

$$v = k(R - r)r^2$$

where k is a constant, R is the normal radius of the trachea, and r is the radius during coughing. What radius will produce the maximum air velocity?

32. The concentration C of a certain chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t}{27 + t^3}.$$

When is the concentration greatest?

33. After a drug is administered to a patient, the drug concentration in the patient's bloodstream over a two-hour period is given by

$$C = 0.29483t + 0.04253t^2 - 0.00035t^3$$

where C is measured in milligrams and t is the time in minutes. Find the open interval on which C is increasing or decreasing.

34. A fast-food restaurant sells x hamburgers to make a profit P given by

$$P = 2.44x - \frac{x^2}{20{,}000} - 5000, \quad 0 \le x \le 35{,}000.$$

Find the open interval on which P is increasing or decreasing.

35. After birth, an infant normally will lose weight for a few days and then start gaining. A model for the average weight W of infants over the first two weeks following birth is

$$W = 0.033t^2 - 0.3974t + 7.3032.$$

Find the open intervals on which W is increasing or decreasing.

36. The electric power P in watts in a direct-current circuit with two resistors R_1 and R_2 connected in series is

$$P = \frac{vR_1R_2}{(R_1 + R_2)^2}$$

where v is the voltage. If v and R_1 are held constant, what resistance R_2 produces maximum power?

37. The resistance R of a certain type of resistor is given by

$$R = \sqrt{0.001T^4 - 4T + 100}$$

where R is measured in ohms and the temperature T is measured in degrees Celsius. What temperature produces a minimum resistance for this type of resistor?

38. Consider the functions $f(x) = x$ and $g(x) = x^3$ on the interval $(0, 1)$.
(a) Prove that $f(x) > g(x)$. [Hint: Show that $h(x) > 0$ where $h = f - g$.]
(b) Sketch the graphs of f and g on the same set of axes.

39. Find a, b, c, and d so that the function given by

$$f(x) = ax^3 + bx^2 + cx + d$$

has a relative minimum at $(0, 0)$ and a relative maximum at $(2, 2)$.

40. Find a, b, and c so that the function given by

$$f(x) = ax^2 + bx + c$$

has a relative maximum at $(5, 20)$ and passes through the point $(2, 10)$.

In Exercises 41–46, assume that f is differentiable for all x. The sign of f' is as follows.

$f'(x) > 0$ on $(-\infty, -4)$
$f'(x) < 0$ on $(-4, 6)$
$f'(x) > 0$ on $(6, \infty)$

In each exercise, supply the appropriate inequality for the indicated value of c.

Function	Sign of $g'(c)$
41. $g(x) = f(x) + 5$	$g'(0)$ ___ 0
42. $g(x) = 3f(x) - 3$	$g'(-5)$ ___ 0
43. $g(x) = -f(x)$	$g'(-6)$ ___ 0
44. $g(x) = -f(x)$	$g'(0)$ ___ 0
45. $g(x) = f(x - 10)$	$g'(0)$ ___ 0
46. $g(x) = f(x - 10)$	$g'(8)$ ___ 0

47. Sketch the graph of a function f such that

$$f'(x) = \begin{cases} > 0, & x < 4 \\ \text{undefined}, & x = 4 \\ < 0 & x > 4 \end{cases}$$

48. A differentiable function f has only one critical number: $x = 5$. Identify the relative extrema of f at the critical number if $f'(4) = -2.5$ and $f'(6) = 3$.

In Exercises 49 and 50, use a computer or graphics calculator (a) to sketch the graph of f and f' on the same coordinate axes over the specified interval, (b) to find the critical numbers of f, and (c) to find the interval(s) on which f' is positive and the interval(s) on which it is negative. Note the behavior of f in relation to the sign of f'.

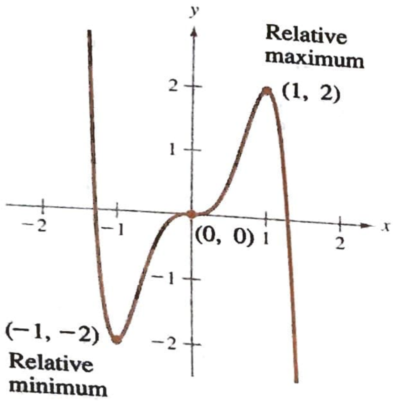
Function	Interval
49. $f(x) = 2x\sqrt{9 - x^2}$	$[-3, 3]$
50. $f(x) = 10(5 - \sqrt{x^2 - 3x + 16})$	$[0, 5]$

51. Prove the second case of Theorem 4.5.
52. Prove the second and third cases of Theorem 4.6.

4.4 Concavity and the Second Derivative Test

Concavity ▪ Points of inflection ▪ The Second Derivative Test

We have already seen that locating the intervals in which a function f increases or decreases helps to determine its graph. In this section we show that, by locating the intervals in which f' increases or decreases, we can determine where the graph of f is *curving upward* or *curving downward*. We refer to this notion of curving upward or downward as **concavity**.



$f(x) = -3x^5 + 5x^3$

FIGURE 4.28

SOLUTION

We begin by finding the critical numbers of f.

$$f'(x) = -15x^4 + 15x^2 = 15x^2(1 - x^2) = 0$$

$$x = -1, 0, 1 \qquad \text{Critical numbers}$$

Using $f''(x) = 30(-2x^3 + x)$, we apply the Second Derivative Test as follows.

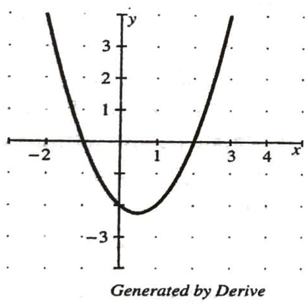
Point	*Sign of f''*		*Conclusion*
$(-1, -2)$	$f''(-1) = 30 > 0$	⇨	Relative minimum
$(1, 2)$	$f''(1) = -30 < 0$	⇨	Relative maximum
$(0, 0)$	$f''(0) = 0$	⇨	Test fails

Since the Second Derivative Test fails at $(0, 0)$, we use the First Derivative Test and observe that f increases to the left and right of $x = 0$. Thus, $(0, 0)$ is neither a relative minimum nor a relative maximum (even though the graph has a horizontal tangent line at this point). The graph of f is shown in Figure 4.28. ▭

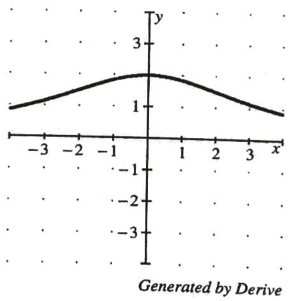
EXERCISES for Section 4.4

In Exercises 1–6, find the open intervals on which the graph of the given function is concave upward and those on which it is concave downward.

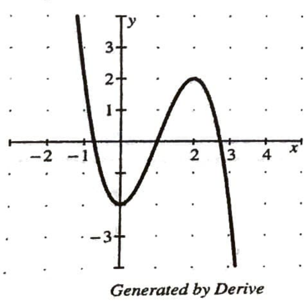
1. $y = x^2 - x - 2$



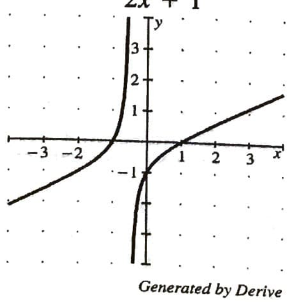
2. $f(x) = \dfrac{24}{x^2 + 12}$



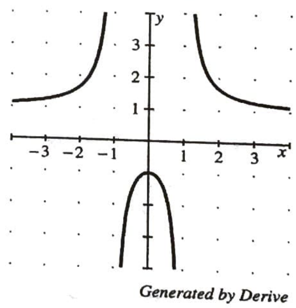
3. $y = -x^3 + 3x^2 - 2$



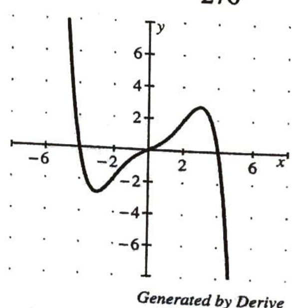
4. $f(x) = \dfrac{x^2 - 1}{2x + 1}$



5. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$



6. $y = \dfrac{-3x^5 + 40x^3 + 135x}{270}$



In Exercises 7–18, identify all relative extrema. Use the Second Derivative Test where applicable.

7. $f(x) = 6x - x^2$

8. $f(x) = x^2 + 3x - 8$

9. $f(x) = (x - 5)^2$

10. $f(x) = -(x - 5)^2$

11. $f(x) = x^3 - 3x^2 + 3$
12. $f(x) = 5 + 3x^2 - x^3$
13. $f(x) = x^4 - 4x^3 + 2$
14. $f(x) = x^3 - 9x^2 + 27x - 26$
15. $f(x) = x^{2/3} - 3$
16. $f(x) = \sqrt{x^2 + 1}$
17. $f(x) = x + \dfrac{4}{x}$
18. $f(x) = \dfrac{x}{x - 1}$

In Exercises 19–34, sketch the graph of the given function and identify all relative extrema and points of inflection.

19. $f(x) = x^3 - 12x$
20. $f(x) = x^3 + 1$
21. $f(x) = x^3 - 6x^2 + 12x - 8$
22. $f(x) = 2x^3 - 3x^2 - 12x + 8$
23. $f(x) = \dfrac{1}{4}x^4 - 2x^2$
24. $f(x) = 2x^4 - 8x + 3$
25. $f(x) = x(x - 4)^3$
26. $f(x) = x^3(x - 4)$
27. $f(x) = x^2 + \dfrac{1}{x^2}$
28. $f(x) = \dfrac{x^2}{x^2 - 1}$
29. $f(x) = x\sqrt{x + 3}$
30. $f(x) = x\sqrt{x + 1}$
31. $f(x) = \dfrac{x}{x^2 - 4}$
32. $f(x) = \dfrac{1}{x^2 - x - 2}$
33. $f(x) = \dfrac{x - 2}{x^2 - 4x + 3}$
34. $f(x) = \dfrac{x + 1}{x^2 + x + 1}$

In Exercises 35 and 36, use a symbolic differentiation utility to analyze the function over the specified interval. (a) Find the first- and second-order derivatives of the function, (b) find any relative extrema and points of inflection, and (c) sketch the graphs of f, f', and f'' on the same coordinate axes and state the relationship between the behavior of f and the signs of f' and f''.

Function	*Interval*
35. $f(x) = 0.2x^2(x - 3)^3$	$[-1, 4]$
36. $f(x) = x^2\sqrt{6 - x^2}$	$[-\sqrt{6}, \sqrt{6}]$

37. Sketch the graph of a function f having the following characteristics.

(a) $f(2) = f(4) = 0$
$f'(x) < 0$ if $x < 3$
$f'(3)$ is undefined
$f'(x) > 0$ if $x > 3$
$f''(x) < 0$, $x \neq 3$

(b) $f(0) = f(2) = 0$
$f'(x) > 0$ if $x < 1$
$f'(1) = 0$
$f'(x) < 0$ if $x > 1$
$f''(x) < 0$

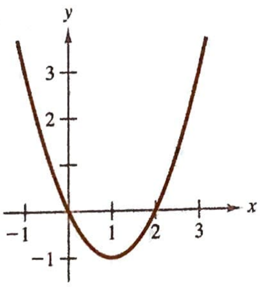
38. Sketch the graph of a function f having the following characteristics.

(a) $f(2) = f(4) = 0$
$f'(x) > 0$ if $x < 3$
$f'(3)$ is undefined
$f'(x) < 0$ if $x > 3$
$f''(x) > 0$, $x \neq 3$

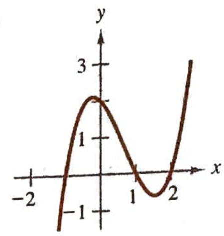
(b) $f(0) = f(2) = 0$
$f'(x) < 0$ if $x < 1$
$f'(1) = 0$
$f'(x) > 0$ if $x > 1$
$f''(x) > 0$

In Exercises 39 and 40, trace the given graph of f. On the same set of axes sketch the graph of f' and f''.

39.



40.



41. Find the inflection points (if any) and sketch the graph of $f(x) = (x - c)^n$ for $n = 1, 2, 3,$ and 4. What conclusion can be made about the value of n and the existence of an inflection point on the graph of f?

42. (a) Sketch the graph of $f(x) = \sqrt[3]{x}$ and identify the inflection point.
(b) Does $f''(x)$ exist at the inflection point?

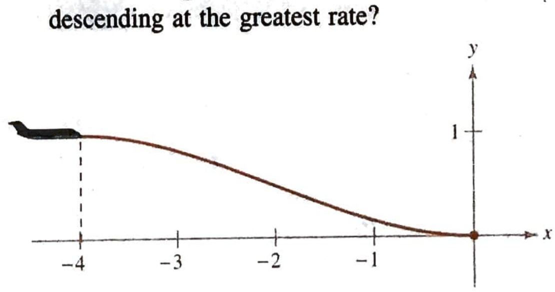
43. Show that the point of inflection of

$$f(x) = x(x - 6)^2$$

lies midway between the relative extrema of f.

44. Prove that a cubic function with three distinct real zeros has a point of inflection whose x-coordinate is the average of the three zeros.

45. A small aircraft starts its descent from an altitude of 1 mile, 4 miles west of the runway (see figure).
(a) Find the cubic polynomial function $f(x) = ax^3 + bx^2 + cx + d$ on the interval $[-4, 0]$, which describes a smooth glide path for the landing.
(b) If the glide path of the plane is described by the function of part (a), when would the plane be descending at the greatest rate?



(For more information on this type of modeling, see the article "How Not to Land at Lake Tahoe!" by Richard Barshinger in the May, 1992, issue of *The American Mathematical Monthly*.)

46. The equation

$$E = \frac{kqx}{(x^2 + a^2)^{3/2}}$$

gives the electric field intensity on the axis of a uniformly charged ring, where q is the total charge on the ring, k is a constant, and a is the radius of the ring. At what value of x is E maximum?

EXERCISES for Section 4.5

In Exercises 1–8, match the given function to one of the graphs (a)–(h), using horizontal asymptotes as an aid.

1. $f(x) = \dfrac{3x^2}{x^2 + 2}$

2. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

3. $f(x) = \dfrac{x}{x^2 + 2}$

4. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

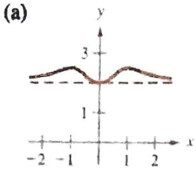
5. $f(x) = \dfrac{-6x}{\sqrt{4x^2 + 5}}$

6. $f(x) = 5 - \dfrac{1}{x^2 + 1}$

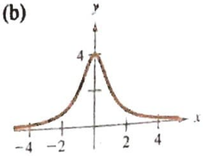
7. $f(x) = \dfrac{4}{x^2 + 1}$

8. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

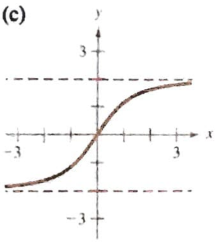
(a)



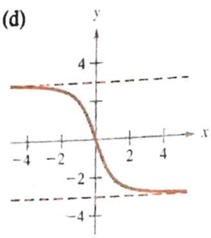
(b)



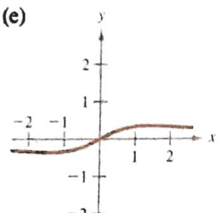
(c)



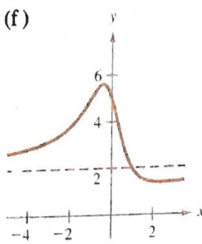
(d)



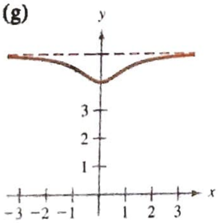
(e)



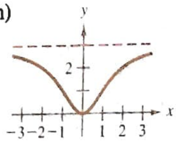
(f)



(g)



(h)



In Exercises 9–24, find the indicated limit.

9. $\displaystyle\lim_{x\to\infty} \frac{2x - 1}{3x + 2}$

10. $\displaystyle\lim_{x\to\infty} \frac{5x^3 + 1}{10x^3 - 3x^2 + 7}$

11. $\displaystyle\lim_{x\to\infty} \frac{x}{x^2 - 1}$

12. $\displaystyle\lim_{x\to\infty} \frac{2x^{10} - 1}{10x^{11} - 3}$

13. $\displaystyle\lim_{x\to-\infty} \frac{5x^2}{x + 3}$

14. $\displaystyle\lim_{x\to\infty} \frac{x^3 - 2x^2 + 3x + 1}{x^2 - 3x + 2}$

15. $\displaystyle\lim_{x\to\infty} \left(2x - \frac{1}{x^2}\right)$

16. $\displaystyle\lim_{x\to\infty} (x + 3)^{-2}$

17. $\displaystyle\lim_{x\to-\infty} \left(\frac{2x}{x - 1} + \frac{3x}{x + 1}\right)$

18. $\displaystyle\lim_{x\to\infty} \left(\frac{2x^2}{x - 1} + \frac{3x}{x + 1}\right)$

19. $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2 - x}}$

20. $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}}$

21. $\displaystyle\lim_{x\to\infty} \frac{2x + 1}{\sqrt{x^2 - x}}$

22. $\displaystyle\lim_{x\to-\infty} \frac{-3x + 1}{\sqrt{x^2 + x}}$

23. $\displaystyle\lim_{x\to\infty} \frac{x^2 - x}{\sqrt{x^4 + x}}$

24. $\displaystyle\lim_{x\to\infty} \frac{2x}{\sqrt{4x^2 + 1}}$

In Exercises 25–28, find the indicated limit. [Hint: Treat the expression as a fraction whose denominator is 1, and rationalize the numerator.]

25. $\displaystyle\lim_{x\to-\infty} (x + \sqrt{x^2 + 3})$

26. $\displaystyle\lim_{x\to\infty} (2x - \sqrt{4x^2 + 1})$

27. $\displaystyle\lim_{x\to\infty} (x - \sqrt{x^2 + x})$

28. $\displaystyle\lim_{x\to-\infty} (3x + \sqrt{9x^2 - x})$

In Exercises 29–40, sketch the graph of the given equation. As a sketching aid, examine each equation for intercepts, symmetry, and asymptotes.

29. $y = \dfrac{2 + x}{1 - x}$

30. $y = \dfrac{x - 3}{x - 2}$

31. $y = \dfrac{x^2}{x^2 + 9}$

32. $y = \dfrac{x^2}{x^2 - 9}$

33. $xy^2 = 4$

34. $x^2y = 4$

35. $y = \dfrac{2x}{1 - x}$

36. $y = \dfrac{2x}{1 - x^2}$

37. $y = 2 - \dfrac{3}{x^2}$

38. $y = 1 + \dfrac{1}{x}$

39. $y = \dfrac{x^3}{\sqrt{x^2 - 4}}$

40. $y = \dfrac{x}{\sqrt{x^2 - 4}}$

In Exercises 41–46, use a symbolic differentiation utility to analyze the graph of the function. Label any extrema and/or asymptotes that exist.

41. $f(x) = 5 - \dfrac{1}{x^2}$

42. $f(x) = \dfrac{x^2}{x^2 - 1}$

43. $f(x) = \dfrac{x}{x^2 - 4}$

44. $f(x) = \dfrac{1}{x^2 - x - 2}$

45. $f(x) = \dfrac{x - 2}{x^2 - 4x + 3}$

46. $f(x) = \dfrac{x + 1}{x^2 + x + 1}$

In Exercises 47 and 48, (a) use a graphing utility to graph f and g on the same coordinate axes, (b) verify algebraically that f and g represent the same function, (c) zoom out sufficiently far so that the graph appears as a line. What is this line? (Note that the points of discontinuity are not readily seen when you zoom out.)

47. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x-3)}$

$g(x) = x + \dfrac{2}{x(x-3)}$

48. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$

$g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

In Exercises 49–52, complete the table and estimate the limit of $f(x)$ as x approaches infinity. Then find the limit analytically and compare your results.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

49. $f(x) = \dfrac{x+1}{x\sqrt{x}}$

50. $f(x) = x - \sqrt{x(x-1)}$

51. $f(x) = 2x - \sqrt{4x^2 + 1}$

52. $f(x) = x^2 - x\sqrt{x(x-1)}$

53. A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is given by $\bar{C} = C/x$. Find the limit of $\bar{C}$ as x approaches infinity.

54. According to the theory of relativity, the mass m of a particle depends on its velocity v. That is,

$$m = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass when the particle is at rest and c is the speed of light. Find the limit of the mass as v approaches c.

55. The efficiency of an internal combustion engine is defined to be

$$\text{efficiency (\%)} = 100\left[1 - \frac{1}{(v_1/v_2)^c}\right]$$

where v_1/v_2 is the ratio of the uncompressed gas to the compressed gas and c is a constant dependent upon the engine design. Find the limit of the efficiency as the compression ratio approaches infinity.

56. Verify that each of the following functions has two horizontal asymptotes.

(a) $f(x) = \dfrac{|x|}{x+1}$ (b) $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$

57. Use a computer or graphics calculator to sketch the graph of $f(x) = 3x/\sqrt{4x^2+1}$, and from the sketch locate any horizontal or vertical asymptotes.

58. Prove that if

$$p(x) = a_nx^n + \cdots + a_1x + a_0$$
$$q(x) = b_mx^m + \cdots + b_1x + b_0$$

then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \\ \pm\infty, & n > m. \end{cases}$$

4.6 A Summary of Curve Sketching

Summary of curve-sketching techniques

It would be difficult to overstate the importance of curve sketching in mathematics. Descartes' introduction of this concept contributed significantly to the rapid advances in calculus that began during the mid-seventeenth century. In the words of Lagrange, "As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace toward perfection."

Today, government, science, industry, business, education, and the social and health sciences all make widespread use of graphs to describe and predict relationships between variables. We have seen, however, that sketching a graph sometimes can require considerable ingenuity.

EXERCISES for Section 4.6

In Exercises 1–26, sketch the graph of the given function. Choose a scale that allows all relative extrema and points of inflection to be identified on the sketch.

1. $y = x^3 - 3x^2 + 3$
2. $y = -\frac{1}{3}(x^3 - 3x + 2)$
3. $y = 2 - x - x^3$
4. $y = x^3 + 3x^2 + 3x + 2$
5. $f(x) = 3x^3 - 9x + 1$
6. $f(x) = (x + 1)(x - 2)(x - 5)$
7. $f(x) = -x^3 + 3x^2 + 9x - 2$
8. $f(x) = \frac{1}{3}(x - 1)^3 + 2$
9. $y = 3x^4 + 4x^3$
10. $y = 3x^4 - 6x^2$
11. $f(x) = x^4 - 4x^3 + 16x$
12. $f(x) = x^4 - 8x^3 + 18x^2 - 16x + 5$
13. $f(x) = x^4 - 4x^3 + 16x - 16$
14. $f(x) = x^5 + 1$
15. $y = x^5 - 5x$
16. $y = (x - 1)^5$
17. $y = |2x - 3|$
18. $y = |x^2 - 6x + 5|$
19. $y = \dfrac{x^2}{x^2 + 3}$
20. $y = \dfrac{x}{x^2 + 1}$
21. $y = x\sqrt{4 - x}$
22. $y = x\sqrt{4 - x^2}$
23. $y = 3x^{2/3} - 2x$
24. $y = 3x^{2/3} - x^2$
25. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}}$
26. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

In Exercises 27–36, sketch the graph of the given function. In each case label the intercepts, relative extrema, points of inflection, and asymptotes.

27. $y = \dfrac{1}{x - 2} - 3$
28. $y = \dfrac{x^2 + 1}{x^2 - 2}$
29. $y = \dfrac{2x}{x^2 - 1}$
30. $y = \dfrac{x^2 - 6x + 12}{x - 4}$
31. $f(x) = \dfrac{x + 2}{x}$
32. $f(x) = x + \dfrac{32}{x^2}$
33. $f(x) = \dfrac{x^2 + 1}{x}$
34. $f(x) = \dfrac{x^3}{x^2 - 1}$
35. $y = \dfrac{x^3}{2x^2 - 8}$
36. $y = \dfrac{2x^2 - 5x + 5}{x - 2}$

In Exercises 37 and 38, use a graphing utility to graph the function.

37. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x}$
38. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$

In Exercises 39 and 40, use a graphing utility to graph the function. Use the graph to determine whether or not it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?

39. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$
40. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

In Exercises 41 and 42, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.

41. $h(x) = \dfrac{6 - 2x}{3 - x}$
42. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

In Exercises 43 and 44, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?

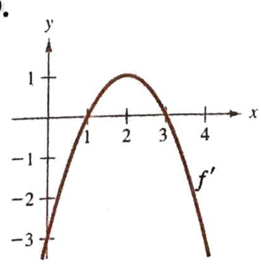
43. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2}$
44. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5}$

In Exercises 45–48, create a function whose graph has the indicated characteristics. (The answer is not unique.)

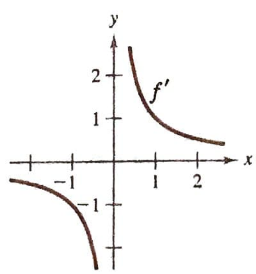
45. Vertical asymptote: $x = 5$
    Horizontal asymptote: $y = 0$
46. Vertical asymptote: $x = -3$
    Horizontal asymptote: None
47. Vertical asymptote: $x = 5$
    Slant asymptote: $y = 3x + 2$
48. Vertical asymptote: $x = 0$
    Slant asymptote: $y = -x$

In Exercises 49–54, use the given graph (of f' or f'') to sketch a graph of the function f. [Hint: The solutions are not unique.]

49.



50.



The feasible domain is $0 \le x \le 1$. Since $dA/dx = [2(\pi + 4)x - 8]/\pi$, the only critical number is $x = 4/(\pi + 4) \approx 0.56$. Therefore, since

REMARK What would the answer be if Example 5 asked for the dimensions needed to enclose the *minimum* total area?

$$A(0) \approx 1.273, \qquad A(0.56) \approx 0.56, \qquad \text{and} \qquad A(1) = 1$$

we conclude that the maximum area occurs when $x = 0$. That is, *all* the wire is used for the circle.

Let's review the primary equations developed in the first five examples. As applications go, these five examples are fairly simple, and yet the resulting primary equations are quite complicated.

$$V = 27x - \frac{x^3}{4} \qquad W = \sqrt{x^2 + 144} + \sqrt{x^2 - 60x + 1684}$$

$$d = \sqrt{x^4 - 3x^2 + 4} \qquad A = \frac{1}{\pi}[(\pi + 4)x^2 - 8x + 4]$$

$$A = 30 + 2x + \frac{72}{x}$$

You must expect that real applications often involve equations that are at least as complicated as these five. Remember that one of the main goals of this course is to learn to use calculus to analyze equations that initially seem formidable.

EXERCISES for Section 4.7

1. Find two positive numbers whose sum is 110 and whose product is a maximum.
   (a) Use paper and pencil to complete six rows of a table like the one below. (The first two rows are shown.) Use the result to guess the two numbers required.

First number, x	Second number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$

   (b) Write the product P as a function of x.
   (c) Use calculus to find the critical number of the function of part (b). Then find the two numbers.
   (d) Use a computer to generate additional rows of the table. Use the table to estimate the solution.
   (e) Use a graphing utility to graph the function in part (b) and estimate the solution from the graph.

In Exercises 2–6, find two positive numbers that satisfy the given requirements.

2. The sum is S and the product is maximum.
3. The product is 192 and the sum is minimum.
4. The product is 192 and the sum of the first plus three times the second is minimum.
5. The second number is the reciprocal of the first and their sum is minimum.
6. The sum of the first and twice the second is 100 and the product is maximum.

In Exercises 7 and 8, find the length and width of a rectangle of maximum area for the given perimeter.

7. Perimeter: 100 feet
8. Perimeter: P units

In Exercises 9 and 10, find the length and width of a rectangle of minimum perimeter for the given area.

9. Area: 64 square feet
10. Area: A square feet

In Exercises 11 and 12, find the point on the graph of the function closest to the given point.

	Function	Point
11.	$f(x) = \sqrt{x}$	$(4, 0)$
12.	$f(x) = x^2$	$\left(2, \frac{1}{2}\right)$

13. A dairy farmer plans to fence in a rectangular pasture adjacent to a river. The pasture must contain 180,000 square meters in order to provide enough grass for the herd. What dimensions would require the least amount of fencing if no fencing is needed along the river?

14. A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?

15. An open box is to be made from a square piece of material, 12 inches on a side, by cutting equal squares from each corner and turning up the sides (see figure). Find the volume of the largest box that can be made.

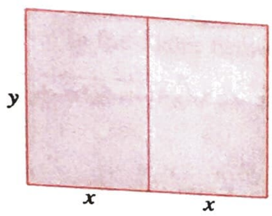


FIGURE FOR 14

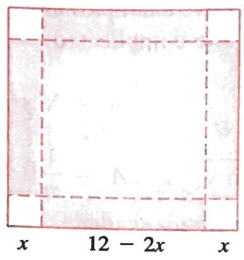


FIGURE FOR 15

16. (a) Solve Exercise 15 given that the square of material is s inches on a side.

(b) If the dimensions of the square piece of material are doubled, how does the volume change?

17. An open box is to be made from a rectangular piece of material by cutting equal squares from each corner and turning up the sides. Find the dimensions of the box of maximum volume if the material has dimensions of 2 feet by 3 feet.

18. A net enclosure for practicing golf is open at one end (see figure). Find the dimensions that require the least amount of netting if the volume of the enclosure is to be $83\frac{1}{3}$ cubic meters.

19. A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). Find the dimensions of a Norman window of maximum area if the total perimeter is 16 feet.

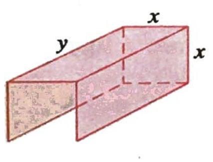


FIGURE FOR 18

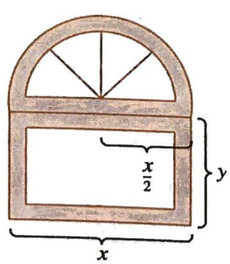


FIGURE FOR 19

20. A physical fitness room consists of a rectangular region with a semicircle on each end. If the perimeter of the room is a 200-meter track, find the dimensions that make the rectangular region as large as possible.

21. A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point (2, 3). Find the vertices of the triangle so that its area is minimum.

22. Find the dimensions of the largest isosceles triangle that can be inscribed in a circle of radius 4 (see figure).

23. A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?

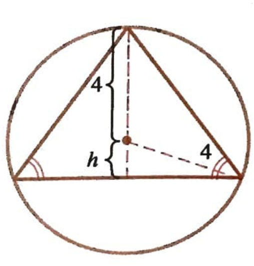


FIGURE FOR 22

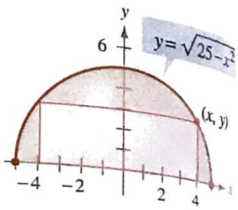


FIGURE FOR 23

24. Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r (see Exercise 23).

25. Find the dimensions of the trapezoid of greatest area that can be inscribed in a semicircle of radius r.

26. A page is to contain 30 square inches of print. The margins at the top and bottom of the page are 2 inches. The margins on each side are only 1 inch. Find the dimensions of the page so that the least paper is used.

27. A right circular cylinder is to be designed to hold 22 cubic inches of soft drink (approximately 12 fluid ounces) and to use a minimum of material in its construction (see figure).

(a) Use paper and pencil to complete four rows of a table like the one below. (The first two rows are shown.) Use the result to guess the minimum amount of material.

Radius	Height	Surface area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.1$

(b) Write the surface area S as a function of radius r.

(c) Use calculus to find the critical number of the function in part (b) and find the dimensions that will yield the minimum surface area.

(d) Use a computer to generate additional rows of the table in part (a) and estimate the minimum surface area from the table.

(e) Use a graphing utility to obtain the graph of the function in part (b) and estimate the required dimensions from the graph.

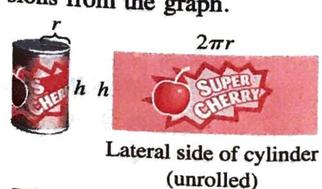


FIGURE FOR 27

28. Use calculus to find the dimensions of the cylinder in Exercise 27 if its volume is V_0 cubic units.
29. A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure). Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)
30. Rework Exercise 29 for a cylindrical package. (The cross sections are circular.)
31. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius r (see figure).

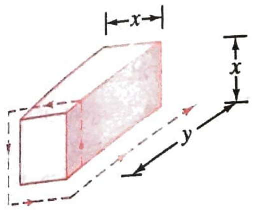


FIGURE FOR 29

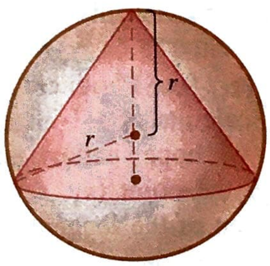


FIGURE FOR 31

32. Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius r.
33. A solid is formed by adjoining two hemispheres to each end of a right circular cylinder. The total volume of the figure is 12 cubic inches. Find the radius of the cylinder that produces the minimum surface area.
34. An industrial tank of the shape described in Exercise 33 must have a volume of 3000 cubic feet. If the construction cost of the hemispherical ends is twice as much per square foot of surface area as the sides, find the dimensions that will minimize cost.
35. The combined perimeter of an equilateral triangle and a square is 10. Find the dimensions of the triangle and square that produce a minimum total area.
36. The combined perimeter of a circle and a square is 16. Find the dimensions of the circle and square that produce a minimum total area.
37. Ten feet of wire is to be used to form an isosceles right triangle and a circle. How much of the wire should be used for the circle if the total area enclosed is to be (a) minimum and (b) maximum?
38. Twenty feet of wire is to be used to form two figures. In each of the following cases, how much should be used for each figure so that the total enclosed area is a maximum?
   (a) equilateral triangle and square
   (b) square and regular pentagon
   (c) regular pentagon and hexagon
39. A wooden beam has a rectangular cross section of height h and width w (see figure). The strength S of the beam is directly proportional to the width and the square of the height. What are the dimensions of the strongest beam that can be cut from a round log of diameter 24 inches? [Hint: $S = kh^2w$, where k is the proportionality constant.]
40. The illumination from a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two light sources of intensities I_1 and I_2 are d units apart. At what point on the line segment joining the two sources is the illumination least?
41. A man is in a boat 2 miles from the nearest point on the coast. He is to go to a point Q, 3 miles down the coast and 1 mile inland (see figure). If he can row at 2 miles per hour and walk at 4 miles per hour, toward what point on the coast should he row in order to reach point Q in the least time?
42. The conditions are the same as in Exercise 41 except that the man can row at 4 miles per hour (see figure). How does this change the solution?

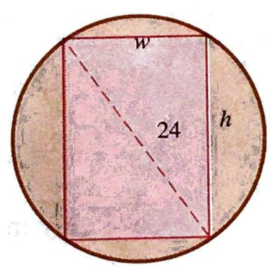


FIGURE FOR 39

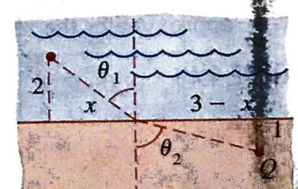


FIGURE FOR 41–42

In Exercises 43 and 44, use a computer or graphics calculator to sketch the graphs of the primary equation and its first derivative in the given applied extrema problem. From the graphs, find the required extrema.

43. Find the dimensions of the rectangle of maximum area that can be inscribed in a semicircle of radius 10.
44. Find the length of the longest pipe that can be carried level around a right-angle corner if the two intersecting corridors are of width 5 feet and 8 feet, respectively.

4.8 Newton's Method

Newton's Method ▪ Algebraic solutions of polynomial equations

In the first seven sections of this chapter, we frequently needed to find the zeros of a function. Until now our functions have been chosen carefully so that elementary algebraic techniques suffice for finding the zeros. For instance,

EXAMPLE 6 Approximating function values

Use differentials to approximate $\sqrt{16.5}$.

SOLUTION

We use the function $f(x) = \sqrt{x}$ and choose $x = 16$. Then $dx = 0.5$, and we obtain

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx$$

$$\sqrt{16.5} \approx \sqrt{16} + \frac{1}{2\sqrt{16}}(0.5) = 4 + \left(\frac{1}{8}\right)\left(\frac{1}{2}\right)$$

$$= 4 + \frac{1}{16} = 4.0625.$$

REMARK The use of differentials to approximate function values diminished with the availability of calculators and computers. Using a calculator, we obtain the value

$$\sqrt{16.5} \approx 4.0620$$

which indicates the accuracy of the differential method in Example 6.

EXERCISES for Section 4.9

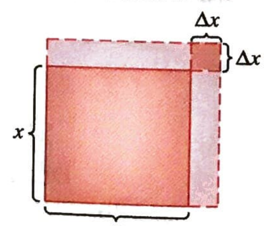
In Exercises 1–10, find the differential dy of the given function.

1. $y = 3x^2 - 4$
2. $y = 2x^{3/2}$
3. $y = 4x^3$
4. $y = 3$
5. $y = \dfrac{x + 1}{2x - 1}$
6. $y = \dfrac{x}{x + 5}$
7. $y = \sqrt{x}$
8. $y = \sqrt{x^2 - 4}$
9. $y = x\sqrt{1 - x^2}$
10. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

In Exercises 11–16, let $x = 2$ and use the given function and value of $\Delta x = dx$ to complete the table.

$dx = \Delta x$	dy	Δy	$\Delta y - dy$	$\dfrac{dy}{\Delta y}$
1.000				
0.500				
0.100				
0.010				
0.001				

11. $y = x - 1$
12. $y = 2x$
13. $y = x^2$
14. $y = \dfrac{1}{x^2}$
15. $y = x^5$
16. $y = \sqrt{x}$
17. The area of a square of side x is given by $A(x) = x^2$.
    (a) Compute dA and ΔA in terms of x and Δx.
    (b) Use the accompanying figure to identify the region whose area is dA.
    (c) Use the accompanying figure to identify the region whose area is $\Delta A - dA$.



18. The measurement of the side of a square is found to be 12 inches, with a possible error of $\frac{1}{64}$ inch. Use differentials to approximate the possible error in computing the area of the square.
19. The measurement of the radius of the end of a log is found to be 14 inches, with a possible error of $\frac{1}{4}$ inch. Use differentials to approximate the possible error in computing the area of the end of the log.

20. The measurement of the edge of a cube is found to be 12 inches, with a possible error of 0.03 inch. Use differentials to approximate the maximum possible error in computing the following.
(a) the volume of the cube
(b) the surface area of the cube

21. The measurement of a side of a square is found to be 15 centimeters.
(a) Approximate the percentage error in computing the area of the square if the possible error in measuring the side is 0.05 centimeters.
(b) Estimate the maximum allowable percentage error in measuring the side if the error in computing the area cannot exceed 2.5%.

22. The measurement of the circumference of a circle is found to be 56 centimeters.
(a) Approximate the percentage error in computing the area of the circle if the possible error in measuring the circumference is 1.2 centimeters.
(b) Estimate the maximum allowable percentage error in measuring the circumference if the error in computing the area cannot exceed 3%.

23. The radius of a sphere is claimed to be 6 inches, with a possible error of 0.02 inch. Use differentials to approximate the maximum possible error in calculating (a) the volume of the sphere and (b) the surface area of the sphere. (c) What is the relative error in parts (a) and (b)?

24. The profit P for a company is given by

$$P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right).$$

Approximate the change in profit as production changes from $x = 115$ to $x = 120$ units. Approximate the percentage change in profit when x changes from $x = 115$ to $x = 120$ units.

25. The period of a pendulum is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the pendulum in feet, g is the acceleration due to gravity, and T is the time in seconds. Suppose that the pendulum has been subjected to an increase in temperature so that the length increases by $\frac{1}{2}$ percent.
(a) Find the approximate percentage change in the period.
(b) Using the result of part (a), find the approximate error in this pendulum clock in one day.

26. Suppose a pendulum of length L is to be used to find the acceleration of gravity at a given point on the earth's surface. Use the formula

$$T = 2\pi\sqrt{\frac{L}{g}}$$

to approximate the percentage error in the value of g if you can identify the period T to within 0.1 percent of its true value.

27. A current of I amps passes through a resistor of R ohms. **Ohms Law** states that the voltage E applied to the resistor is given by

$$E = IR.$$

If the voltage is constant, show that the magnitude of the relative error in R caused by a change in I is equal in magnitude to the relative error in I.

28. The cost in dollars of removing p% of the air pollutants in the stack emission of a utility company that burns coal to generate electricity is

$$C = \frac{80{,}000p}{100 - p}, \quad 0 \le p < 100.$$

Use differentials to approximate the increase in cost if the government requires the utility company to remove 2% more of the pollutants and p is currently
(a) 40% (b) 75%

29. Show that if $y = f(x)$ is a differentiable function, then

$$\Delta y - dy = \varepsilon\,\Delta x$$

where $\varepsilon \to 0$ as $\Delta x \to 0$.

4.10 Business and Economics Applications

Marginals ▪ Demand function

In Section 3.7 we discussed one of the most common ways to measure change—with respect to time. In this section we study some important rates of change in economics that are not measured with respect to time. For example, economists refer to **marginal profit, marginal revenue,** and **marginal cost** as the rates of change of the profit, revenue, and cost with respect to the number of units produced or sold.

We begin with a summary of some basic terms and formulas.