REMARK In Example 8, note that the position function has the form

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$

where $g = -32$, v_0 is the initial velocity, and s_0 is the initial height, as presented earlier in Section 3.2.

Before you begin the exercise set for this section, be sure you realize that one of the most important steps in integration is *rewriting the integrand* in a form that fits the basic integration rules. To further illustrate this point, we list several additional examples in Table 5.1.

TABLE 5.1

Given	*Rewrite*	*Integrate*	*Simplify*
$\int \frac{2}{\sqrt{x}}\,dx$	$2\int x^{-1/2}\,dx$	$2\left(\frac{x^{1/2}}{1/2}\right) + C$	$4x^{1/2} + C$
$\int (t^2 + 1)^2\,dt$	$\int (t^4 + 2t^2 + 1)\,dt$	$\frac{t^5}{5} + 2\left(\frac{t^3}{3}\right) + t + C$	$\frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$
$\int \frac{x^3 + 3}{x^2}\,dx$	$\int (x + 3x^{-2})\,dx$	$\frac{x^2}{2} + 3\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{1}{2}x^2 - \frac{3}{x} + C$
$\int \sqrt[3]{x}(x - 4)\,dx$	$\int (x^{4/3} - 4x^{1/3})\,dx$	$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$	$\frac{3}{7}x^{4/3}(x - 7) + C$

EXERCISES for Section 5.1

In Exercises 1–6, complete the table using Table 5.1 as a model.

Given	*Rewrite*	*Integrate*	*Simplify*
1. $\int \sqrt[3]{x}\,dx$			
2. $\int \frac{1}{x^2}\,dx$			
3. $\int \frac{1}{x\sqrt{x}}\,dx$			
4. $\int x(x^2 + 3)\,dx$			
5. $\int \frac{1}{2x^3}\,dx$			
6. $\int \frac{1}{(2x)^3}\,dx$			

In Exercises 7–26, evaluate the indefinite integral and check your result by differentiation.

7. $\int (x^3 + 2)\,dx$
8. $\int (x^2 - 2x + 3)\,dx$
9. $\int (x^{3/2} + 2x + 1)\,dx$
10. $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right)dx$
11. $\int \sqrt[3]{x^2}\,dx$
12. $\int (\sqrt[4]{x^3} + 1)\,dx$
13. $\int \frac{1}{x^3}\,dx$
14. $\int \frac{1}{x^4}\,dx$
15. $\int \frac{1}{4x^2}\,dx$
16. $\int (2x + x^{-1/2})\,dx$
17. $\int \frac{x^2 + x + 1}{\sqrt{x}}\,dx$
18. $\int \frac{x^2 + 1}{x^2}\,dx$
19. $\int (x + 1)(3x - 2)\,dx$
20. $\int (2t^2 - 1)^2\,dt$

21. $\int \frac{t^2 + 2}{t^2}\,dt$

22. $\int (1 - 2y + 3y^2)\,dy$

23. $\int y^2\sqrt{y}\,dy$

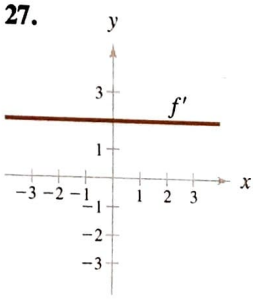
24. $\int (1 + 3t)t^2\,dt$

25. $\int dx$

26. $\int 3\,dt$

In Exercises 27–30, the graph of the derivative of a function is given. Sketch the graphs of *two* functions that have the given derivative. (There is more than one correct answer.)

27.



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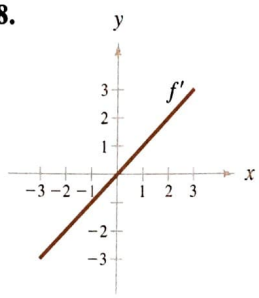

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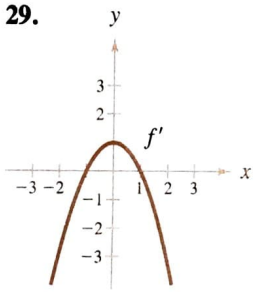

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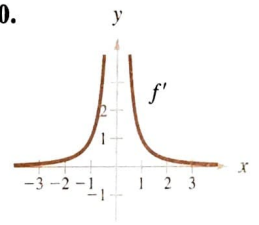
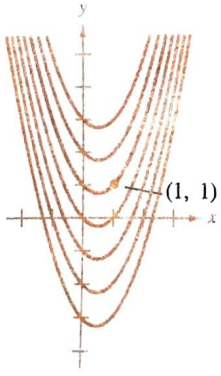
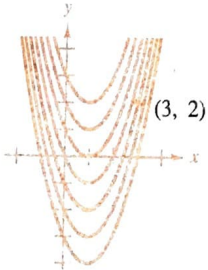


In Exercises 31–34, find the equation of the curve, given the derivative and the indicated point on the curve.

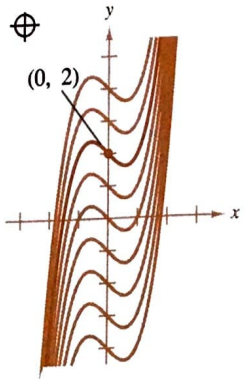
31. $\frac{dy}{dx} = 2x - 1$



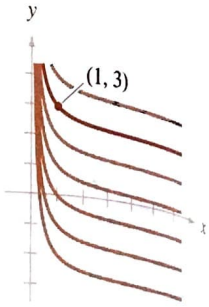
32. $\frac{dy}{dx} = 2(x - 1)$



33. $\frac{dy}{dx} = 3x^2 - 1$



34. $\frac{dy}{dx} = -\frac{1}{x^2}, x > 0$



In Exercises 35–38, find $y = f(x)$ satisfying the given conditions.

35. $f''(x) = 2, f'(2) = 5, f(2) = 10$
36. $f''(x) = x^2, f'(0) = 6, f(0) = 3$
37. $f''(x) = x^{-3/2}, f'(4) = 2, f(0) = 0$
38. $f''(x) = x^{-3/2}, f'(1) = 2, f(9) = -4$

In Exercises 39–43, use $a(t) = -32$ ft/s^2 as the acceleration due to gravity. (Neglect air resistance.)

39. An object is dropped from a balloon that is stationary at 1600 feet above the ground. Express its height above the ground as a function of t. How long does it take the object to reach the ground?

40. A ball is thrown vertically upward from the ground with an initial velocity of 60 feet per second. How high will the ball go?

41. With what initial velocity must an object be thrown upward (from ground level) to reach a maximum height of 550 feet (approximate height of the Washington Monument)?

42. Show that the height above the ground of an object thrown upward from a point s_0 feet above the ground with an initial velocity of v_0 feet per second is given by the function

$$f(t) = -16t^2 + v_0t + s_0.$$

43. A balloon, rising vertically with a velocity of 16 feet per second, releases a sandbag at the instant when the balloon is 64 feet above the ground.
(a) How many seconds after its release will the bag strike the ground?
(b) With what velocity will it reach the ground?

44. Assume that a fully loaded plane starting from rest has a constant acceleration while moving down the runway.

Now, from the Mean Value Theorem for Integrals, we know there exists a number c in the interval $[x, x + \Delta x]$ such that the integral in the above expression is equal to $f(c)\Delta x$. Moreover, since $x \le c \le x + \Delta x$, it follows that $c \to x$ as $\Delta x \to 0$. Thus, we have

$$F'(x) = \lim_{\Delta x \to 0} \left[\frac{1}{\Delta x} f(c)\Delta x \right] = \lim_{\Delta x \to 0} f(c) = f(x).$$

REMARK Using the area model for definite integrals, we can view the approximation

$$f(x)\Delta x \approx \int_x^{x+\Delta x} f(t)\, dt$$

as saying that the area of the rectangle of height $f(x)$ and width Δx is approximately equal to the area of the region lying between the graph of f and the x-axis on the interval $[x, x + \Delta x]$, as shown in Figure 5.29.

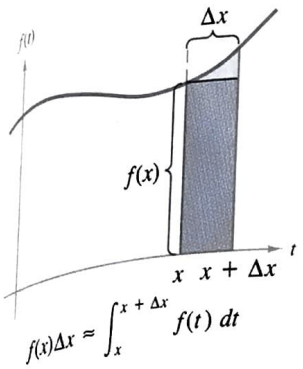


FIGURE 5.29

Note that the Second Fundamental Theorem of Calculus tells us that if a function is continuous, then we can be sure that it has an antiderivative. This antiderivative need not, however, be an elementary function. (Recall the discussion of elementary functions in Section 1.5.)

EXAMPLE 6 Applying the Second Fundamental Theorem of Calculus

Evaluate

$$\frac{d}{dx} \int_0^x \sqrt{t^2 + 1}\, dt.$$

SOLUTION

Note that $f(t) = \sqrt{t^2 + 1}$ is continuous on the entire real line. Thus, using the Second Fundamental Theorem of Calculus, we can write

$$\frac{d}{dx} \int_0^x \sqrt{t^2 + 1}\, dt = \sqrt{x^2 + 1}.$$

XERCISES for Section 5.4

Exercises 1–24, evaluate the definite integral.

1. $\int_0^1 2x\, dx$
2. $\int_2^7 3\, dv$
3. $\int_{-1}^0 (x - 2)\, dx$
4. $\int_2^5 (-3v + 4)\, dv$
5. $\int_{-1}^1 (t^2 - 2)\, dt$
6. $\int_0^3 (3x^2 + x - 2)\, dx$
7. $\int_0^1 (2t - 1)^2\, dt$
8. $\int_{-1}^1 (t^3 - 9t)\, dt$
9. $\int_1^2 \left(\frac{3}{x^2} - 1\right) dx$
10. $\int_0^1 (3x^3 - 9x + 7)\, dx$
11. $\int_1^2 (5x^4 + 5)\, dx$
12. $\int_{-3}^3 v^{1/3}\, dv$
13. $\int_{-1}^1 (\sqrt[3]{t} - 2)\, dt$
14. $\int_1^2 \sqrt{\frac{2}{x}}\, dx$
15. $\int_1^4 \frac{u - 2}{\sqrt{u}}\, du$
16. $\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du$
17. $\int_0^1 \frac{x - \sqrt{x}}{3} dx$
18. $\int_0^2 (2 - t)\sqrt{t}\, dt$

19. $\int_{-1}^{0} (t^{1/3} - t^{2/3})\, dt$

20. $\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}}\, dx$

21. $\int_{-1}^{1} |x|\, dx$

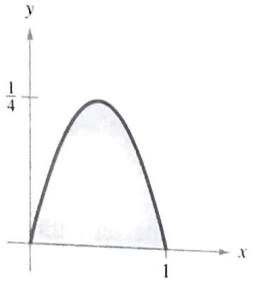
22. $\int_{0}^{3} |2x - 3|\, dx$

23. $\int_{0}^{4} |x^2 - 4x + 3|\, dx$

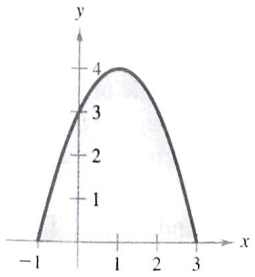
24. $\int_{-1}^{1} |x^3|\, dx$

In Exercises 25–30, determine the area of the indicated region.

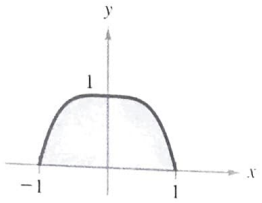
25. $y = x - x^2$



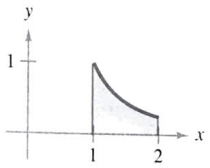
26. $y = -x^2 + 2x + 3$



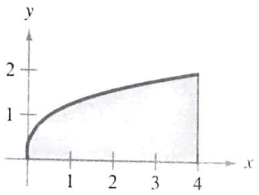
27. $y = 1 - x^4$



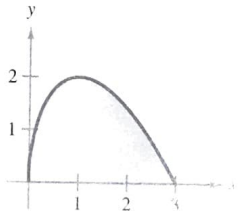
28. $y = \frac{1}{x^2}$



29. $y = \sqrt[3]{2x}$



30. $y = (3 - x)\sqrt{x}$



In Exercises 31–34, find the area of the region bounded by the graphs of the given equations.

31. $y = 3x^2 + 1,\ x = 0,\ x = 2,\ y = 0$
32. $y = 1 + \sqrt{x},\ x = 0,\ x = 4,\ y = 0$
33. $y = x^3 + x,\ x = 2,\ y = 0$
34. $y = -x^2 + 3x,\ y = 0$

In Exercises 35–38, find the values of c guaranteed by the *Mean Value Theorem for Integrals* for the given function over the specified interval.

Function	*Interval*
35. $f(x) = x^3$	$[0, 2]$
36. $f(x) = \frac{9}{x^3}$	$[1, 3]$
37. $f(x) = -x^2 + 4x$	$[0, 3]$
38. $f(x) = \sqrt{x}$	$[1, 9]$

In Exercises 39–42, sketch the graph of the given function over the specified interval. Find the average value of the function over the interval and all values of x where the function equals its average value.

Function	*Interval*
39. $f(x) = 4 - x^2$	$[-2, 2]$
40. $f(x) = \frac{x^2 + 1}{x^2}$	$\left[\frac{1}{2}, 2\right]$
41. $f(x) = x - 2\sqrt{x}$	$[0, 4]$
42. $f(x) = \frac{1}{(x - 3)^2}$	$[0, 2]$

In Exercises 43–48, (a) integrate to find F as a function of x and (b) demonstrate the Second Fundamental Theorem of Calculus by differentiating the result of part (a).

43. $F(x) = \int_{0}^{x} (t + 2)\, dt$

44. $F(x) = \int_{0}^{x} t(t^2 + 1)\, dt$

45. $F(x) = \int_{8}^{x} \sqrt[3]{t}\, dt$

46. $F(x) = \int_{4}^{x} \sqrt{t}\, dt$

47. $F(x) = \int_{1}^{x} \frac{1}{t^2}\, dt$

48. $F(x) = \int_{0}^{x} t^{3/2}\, dt$

In Exercises 49–52, use the Second Fundamental Theorem of Calculus to find $F'(x)$.

49. $F(x) = \int_{-2}^{x} (t^2 - 2t + 5)\, dt$

50. $F(x) = \int_{1}^{x} \sqrt[4]{t}\, dt$

51. $F(x) = \int_{-1}^{x} \sqrt{t^4 + 1}\, dt$

52. $F(x) = \int_{1}^{x} \frac{t^2}{t^2 + 1}\, dt$

53. The volume V in liters of air in the lungs during a 5-second respiratory cycle is approximated by the model

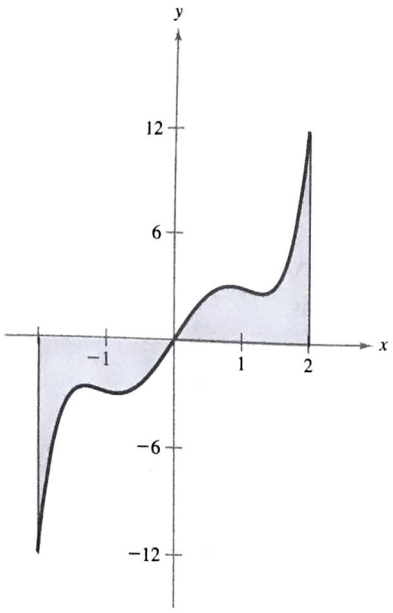
$$V = 0.1729t + 0.1522t^2 - 0.0374t^3$$

where t is the time in seconds. Approximate the average volume of air in the lungs during one cycle.

54. The velocity v of the flow of blood at a distance r from the central axis of an artery of radius R is given by

$$v = k(R^2 - r^2)$$

where k is the constant of proportionality. Find the average rate of flow of blood along a radius of the artery. (Use zero and R as the limits of integration.)



$f(x) = x^5 - 4x^3 + 6x$

FIGURE 5.32

EXAMPLE 10 Integration of an odd function

Evaluate

$$\int_{-2}^{2} (x^5 - 4x^3 + 6x)\,dx.$$

SOLUTION

By letting $f(x) = x^5 - 4x^3 + 6x$, we have

$$f(-x) = (-x)^5 - 4(-x)^3 + 6(-x) = -x^5 + 4x^3 - 6x = -f(x).$$

Thus, f is an odd function, and since $[-2, 2]$ is symmetric about the origin, we can apply Theorem 5.17 to conclude that

$$\int_{-2}^{2} (x^5 - 4x^3 + 6x)\,dx = 0.$$

□

REMARK From Figure 5.32, we see that the two regions on either side of the y-axis have the same area. However, since one lies below the x-axis and one lies above, integration produces a cancellation effect. (We will say more about finding the area of a region below the x-axis in Section 6.1.)

EXERCISES for Section 5.5

In Exercises 1–4, complete the table by identifying u and du for the given integral.

$\int f(g(x))g'(x)\,dx$	$u = g(x)$	$du = g'(x)\,dx$

1. $\int (5x^2 + 1)^2(10x)\,dx$
2. $\int x^2\sqrt{x^3 + 1}\,dx$
3. $\int \dfrac{x}{\sqrt{x^2 + 1}}\,dx$
4. $\int (x^3 + 3)\,3x^2\,dx$

In Exercises 5–28, evaluate the indefinite integral and check the result by differentiation.

5. $\int (1 + 2x)^4(2)\,dx$
6. $\int (x^2 - 1)^3(2x)\,dx$
7. $\int \sqrt{9 - x^2}(-2x)\,dx$
8. $\int (1 - 2x^2)^3(-4x)\,dx$
9. $\int x^2(x^3 - 1)^4\,dx$
10. $\int x(4x^2 + 3)^3\,dx$
11. $\int 5x\sqrt[3]{1 - x^2}\,dx$
12. $\int u^3\sqrt{u^4 + 2}\,du$
13. $\int \dfrac{x^2}{(1 + x^3)^2}\,dx$
14. $\int \dfrac{x^2}{(16 - x^3)^2}\,dx$
15. $\int \dfrac{4x}{\sqrt{16 - x^2}}\,dx$
16. $\int \dfrac{10x^2}{\sqrt{1 + x^3}}\,dx$
17. $\int \dfrac{x + 1}{(x^2 + 2x - 3)^2}\,dx$
18. $\int \dfrac{x - 4}{\sqrt{x^2 - 8x + 1}}\,dx$
19. $\int \left(1 + \dfrac{1}{t}\right)^3\left(\dfrac{1}{t^2}\right)dt$
20. $\int \dfrac{1}{(3x)^2}\,dx$
21. $\int \dfrac{1}{\sqrt{2x}}\,dx$
22. $\int \dfrac{1}{2\sqrt{x}}\,dx$
23. $\int \dfrac{x^2 + 3x + 7}{\sqrt{x}}\,dx$
24. $\int \dfrac{t + 2t^2}{\sqrt{t}}\,dt$
25. $\int t^2\left(t - \dfrac{2}{t}\right)dt$
26. $\int \left(\dfrac{t^3}{3} + \dfrac{1}{4t^2}\right)dt$
27. $\int (9 - y)\sqrt{y}\,dy$
28. $\int 2\pi y(8 - y^{3/2})\,dy$

In Exercises 29–38, evaluate the indefinite integral by the method shown in Example 5.

29. $\int x\sqrt{x + 2}\,dx$
30. $\int x\sqrt{2x + 1}\,dx$
31. $\int x^2\sqrt{1 - x}\,dx$
32. $\int x^3\sqrt{x + 2}\,dx$
33. $\int \dfrac{x^2 - 1}{\sqrt{2x - 1}}\,dx$
34. $\int \dfrac{2x - 1}{\sqrt{x + 3}}\,dx$

35. $\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$

36. $\int t\sqrt[3]{t-4}\,dt$

37. $\int \frac{x}{\sqrt{2x+1}}\,dx$

38. $\int (x+1)\sqrt{2-x}\,dx$

In Exercises 39–50, evaluate the definite integral.

39. $\int_{-1}^{1} x(x^2+1)^3\,dx$

40. $\int_{0}^{1} x\sqrt{1-x^2}\,dx$

41. $\int_{0}^{4} \frac{1}{\sqrt{2x+1}}\,dx$

42. $\int_{0}^{2} \frac{x}{\sqrt{1+2x^2}}\,dx$

43. $\int_{1}^{9} \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

44. $\int_{0}^{2} x\sqrt[3]{4+x^2}\,dx$

45. $\int_{1}^{2} (x-1)\sqrt{2-x}\,dx$

46. $\int_{0}^{4} \frac{x}{\sqrt{2x+1}}\,dx$

47. $\int_{3}^{7} x\sqrt{x-3}\,dx$

48. $\int_{0}^{1} \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

49. $\int_{0}^{7} x\sqrt[3]{x+1}\,dx$

50. $\int_{-2}^{6} x^2\sqrt[3]{x+2}\,dx$

51. Use the fact that

$$\int_0^2 x^2\,dx = \frac{8}{3}$$

to evaluate the following definite integrals without using the Fundamental Theorem of Calculus.

(a) $\int_{-2}^{0} x^2\,dx$ (b) $\int_{-2}^{2} x^2\,dx$

(c) $\int_{0}^{2} -x^2\,dx$ (d) $\int_{-2}^{0} 3x^2\,dx$

52. Find the equation of the function f whose graph passes through the point $\left(0, \frac{7}{3}\right)$ and whose derivative is $f'(x) = x\sqrt{1-x^2}$.

53. A lumber company is seeking a model that yields the average weight loss W per log as a function of the number of days of drying time t. The model is to be reliable up to 100 days after the log is cut. Based on the weight loss during the first 30 days, it was determined that

$$\frac{dW}{dt} = \frac{12}{\sqrt{16t+9}}.$$

(a) Find W as a function of t. Note that no weight loss occurs until the tree is cut.

(b) Find the total weight loss after 100 days.

54. The marginal cost for a certain commodity has been determined to be

$$\frac{dC}{dx} = \frac{12}{\sqrt[3]{12x+1}}.$$

(a) Find the cost function if $C = 100$ when $x = 13$.

(b) Graph the marginal cost function and the cost function on the same set of axes.

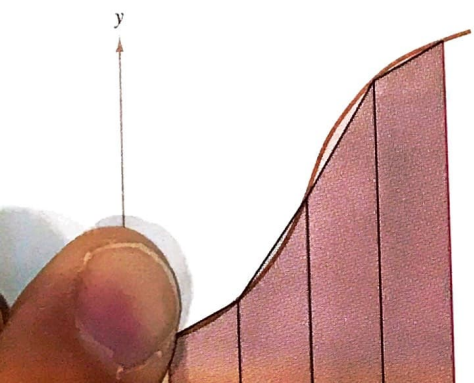
5.6 Numerical Integration

The Trapezoidal Rule ▪ Simpson's Rule

Occasionally, we encounter functions for which we cannot find antiderivatives. Of course, that may be due to a lack of cleverness on our part. On the other hand, some elementary functions simply do not possess antiderivatives that are elementary functions. For example, there is no elementary function that has either of the following functions as its derivative.

$$\sqrt[3]{x}\sqrt{1-x} \qquad \sqrt{1-x^3}$$

If we wish to evaluate a definite integral involving a function whos antiderivative we cannot find, then the Fundamental Theorem of Calculu cannot be applied, and we must resort to an approximation technique. W describe two such techniques in this section.



The Trapezoidal Rule

One way to approximate a definite integral is by the use of n trapezoids shown in Figure 5.33. In the development of this method, we assume th and positive on the interval $[a, b]$, and thus the definite inte by the graph of f

SOLUTION

Since the graph of the pre-1979 model lies above the post-1979 graph on the interval [9, 16], the amount of gasoline saved is given by the following integral.

$$\int_9^{16} \overbrace{[(0.000433t^2 + 0.0962t + 2.76)}^{f(t)} - \overbrace{(-0.00831t^2 + 0.152t + 2.81)]}^{g(t)}\, dt$$

$$= \int_9^{16} (0.008743t^2 - 0.0558t - 0.05)\, dt$$

$$= \left[\frac{0.008743t^3}{3} - \frac{0.0558t^2}{2} - 0.05t\right]_9^{16}$$

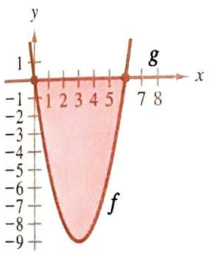
$$\approx 4.58 \text{ billion barrels}$$

Therefore, approximately 4.58 billion barrels of fuel were saved. (At 42 gallons per barrel, about 200 billion gallons were saved!) ▭

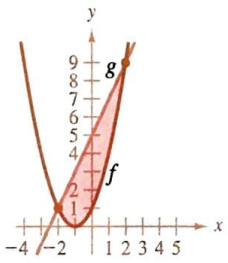
EXERCISES for Section 6.1

In Exercises 1–6, find the area of the given region.

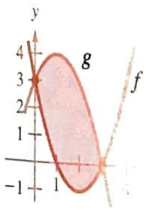
1. $f(x) = x^2 - 6x$
$g(x) = 0$



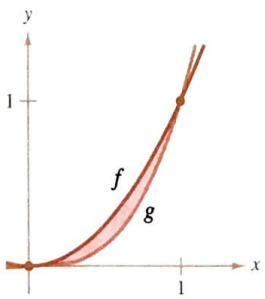
2. $f(x) = x^2 + 2x + 1$
$g(x) = 2x + 5$



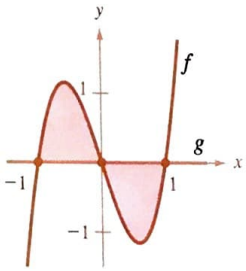
3. $f(x) = x^2 - 4x + 3$
$g(x) = -x^2 + 2x + 3$



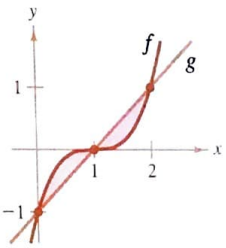
4. $f(x) = x^2$
$g(x) = x^3$



5. $f(x) = 3(x^3 - x)$
$g(x) = 0$



6. $f(x) = (x - 1)^3$
$g(x) = x - 1$



In Exercises 7 and 8, determine which value best approximates the area of the region bounded by the graphs of f and g. (Make your selection on the basis of a sketch of the region and *not* by performing any calculations.)

7. $f(x) = x + 1$, $g(x) = (x - 1)^2$
(a) -2 (b) 2 (c) 10 (d) 4 (e) 8

8. $f(x) = 2 - \frac{1}{2}x$, $g(x) = 2 - \sqrt{x}$
(a) 1 (b) 6 (c) -3 (d) 3 (e) 4

In Exercises 9–26, sketch the region bounded by the graphs of the given functions and find the area of the region.

9. $f(x) = x^2 - 4x$, $g(x) = 0$
10. $f(x) = 3 - 2x - x^2$, $g(x) = 0$
11. $f(x) = x^2 + 2x + 1$, $g(x) = 3x + 3$
12. $f(x) = -x^2 + 4x + 2$, $g(x) = x + 2$
13. $y = x$, $y = 2 - x$, $y = 0$
14. $y = \frac{1}{x^2}$, $y = 0$, $x = 1$, $x = 5$
15. $f(x) = 3x^2 + 2x$, $g(x) = 8$
16. $f(x) = x(x^2 - 3x + 3)$, $g(x) = x^2$
17. $f(x) = x^3 - 2x + 1$, $g(x) = -2x$, $x = 1$
18. $f(x) = \sqrt[3]{x}$, $g(x) = x$
19. $f(x) = \sqrt{3x} + 1$, $g(x) = x + 1$
20. $f(x) = x^2 + 5x - 6$, $g(x) = 6x - 6$
21. $y = x^2 - 4x + 3$, $y = 3 + 4x - x^2$
22. $y = x^4 - 2x^2$, $y = 2x^2$
23. $f(y) = y^2$, $g(y) = y + 2$
24. $f(y) = y(2 - y)$, $g(y) = -y$
25. $f(y) = y^2 + 1$, $g(y) = 0$, $y = -1$, $y = 2$
26. $f(y) = \frac{y}{\sqrt{16 - y^2}}$, $g(y) = 0$, $y = 3$

In Exercises 27 and 28, use a symbolic integration utility to graph the region bounded by the graphs of the functions and find the area of the region.

27. $f(x) = x^4$, $g(x) = 3x + 4$
28. $f(x) = x^6$, $g(x) = x + 2$

In Exercises 29–32, use integration to find the area of the triangle having the given vertices.

29. (0, 0), (4, 0), (4, 4)
30. (0, 0), (4, 0), (6, 4)
31. (0, 0), $(a, 0)$, (b, c)
32. (2, −3), (4, 6), (6, 1)

In Exercises 33 and 34, find b so that the line $y = b$ divides the region bounded by the graphs of the two equations into two regions of equal area.

33. $y = 9 - x^2$, $y = 0$
34. $y = 9 - |x|$, $y = 0$

35. The graphs of $y = x^4 - 2x^2 + 1$ and $y = 1 - x^2$ intersect at three points. However, the area between the curves *can* be found by a single integral. Explain why this is so, and write an integral for this area.

36. The area of the region bounded by the graphs of $y = x^3$ and $y = x$ *cannot* be found by the single integral

$$\int_{-1}^{1} (x^3 - x)\, dx.$$

Explain why this is so. Use symmetry to write a single integral that does represent the area.

In Exercises 37 and 38, find the area of the region bounded by the graph of the function and the tangent line to the graph at the specified point.

Function	*Point*
37. $f(x) = x^3$	(1, 1)
38. $f(x) = \sqrt[3]{x - 1}$	(2, 1)

In Exercises 39 and 40, use a computer or calculator and Simpson's Rule (with $n = 4$) to approximate the area of the region bounded by the graphs of the given equations.

39. $y = \sqrt{1 + x^3}$, $y = \frac{1}{2}x + 2$, $x = 0$
40. $y = \sqrt{x + x^2}$, $y = 0$, $x = 0$, $x = 1$

In Exercises 41 and 42, evaluate the given limit and sketch the graph of the region whose area is given by the limit.

41. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i - x_i^2)\, \Delta x$
where $x_i = i/n$ and $\Delta x = 1/n$

42. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (4 - x_i^2)\, \Delta x$
where $x_i = -2 + (4i/n)$ and $\Delta x = 4/n$

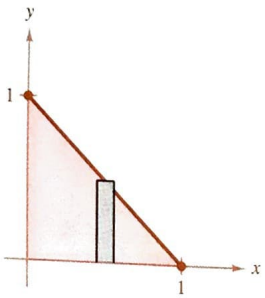
In Exercises 43 and 44, two models R_1 and R_2 are given for revenue (in billions of dollars) for a large corporation. The model R_1 gives projected annual revenues from 1990 to 1995, with $t = 0$ corresponding to 1990, and R_2 gives projected revenues if there is a decrease in growth of corporate sales over the period. Approximate the total reduction in revenue if corporate sales are actually closer to the model R_2.

43. $R_1 = 7.21 + 0.58t$
$R_2 = 7.21 + 0.45t$
44. $R_1 = 7.21 + 0.26t + 0.02t^2$
$R_2 = 7.21 + 0.1t + 0.01t^2$

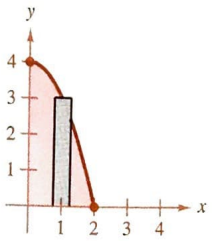
EXERCISES for Section 6.2

In Exercises 1–8, find the volume of the solid formed by revolving the given region about the x-axis.

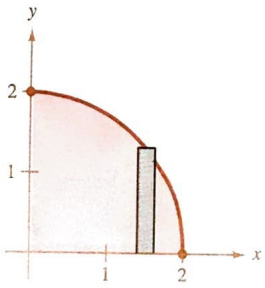
1. $y = -x + 1$



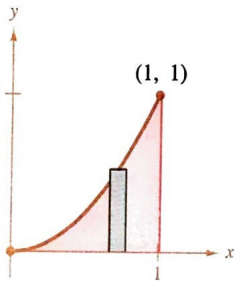
2. $y = 4 - x^2$



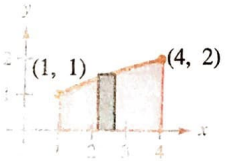
3. $y = \sqrt{4 - x^2}$



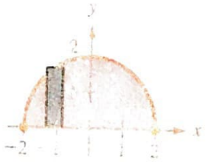
4. $y = x^2$



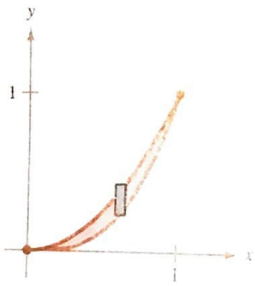
5. $y = \sqrt{x}$



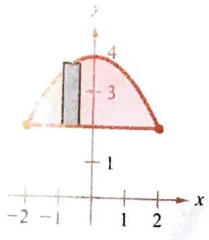
6. $y = \sqrt{4 - x^2}$



7. $y = x^2, y = x^3$

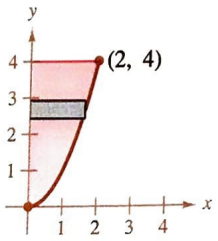


8. $y = 2, y = 4 - \frac{x^2}{2}$

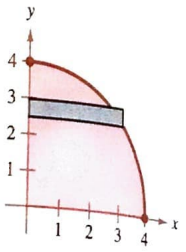


In Exercises 9–12, find the volume of the solid formed by revolving the given region about the y-axis.

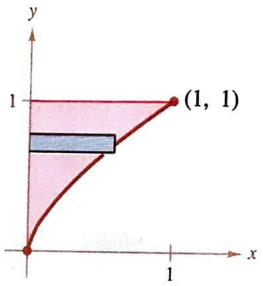
9. $y = x^2$



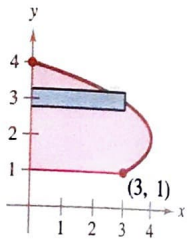
10. $y = \sqrt{16 - x^2}$



11. $y = x^{2/3}$



12. $x = -y^2 + 4y$



In Exercises 13–16, find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the indicated lines.

13. $y = \sqrt{x}, y = 0, x = 4$

(a) the x-axis (b) the y-axis

(c) the line $x = 4$ (d) the line $x = 6$

14. $y = 2x^2, y = 0, x = 2$

(a) the y-axis (b) the x-axis

(c) the line $y = 8$ (d) the line $x = 2$

15. $y = x^2, y = 4x - x^2$

(a) the x-axis (b) the line $y = 6$

16. $y = 6 - 2x - x^2, y = x + 6$

(a) the x-axis (b) the line $y = 3$

In Exercises 17–20, find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the x-axis.

17. $y = x\sqrt{4 - x^2}, y = 0$

18. $y = 2x^{2/3} - x, y = 0$

19. $y = \frac{1}{x}, y = 0, x = 1, x = 4$

20. $y = \frac{3}{x + 1}, y = 0, x = 0, x = 8$

In Exercises 21–24, find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the line $y = 4$.

21. $y = x, y = 3, x = 0$
22. $y = x^2, y = 4$
23. $y = \frac{1}{x^2}, y = 0, x = 1, x = 4$
24. $y = \sqrt{x}, y = 0, 0 \le x \le 4$

In Exercises 25–28, find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the line $x = 6$.

25. $y = x, y = 0, y = 4, x = 6$
26. $y = 6 - x, y = 0, y = 4, x = 0$
27. $x = y^2, x = 4$
28. $xy^2 = 12, y = 2, y = 6, x = 6$

29. The region bounded by the parabola $y = 4x - x^2$ and the x-axis is revolved about the x-axis. Find the volume of the resulting solid.
30. If the equation of the parabola in Exercise 29 were changed to $y = 4 - x^2$, would the volume of the solid generated be different? Why or why not?
31. The upper half of the ellipse $9x^2 + 25y^2 = 225$ is revolved about the x-axis to form a prolate spheroid (shaped like a football). Find the volume of the spheroid.
32. The right half of the ellipse $9x^2 + 25y^2 = 225$ is revolved about the y-axis to form an oblate spheroid (shaped like an M&M candy). Find the volume of the spheroid.
33. If the portion of the line $y = \frac{1}{2}x$ lying in the first quadrant is revolved about the x-axis, a cone is generated. Find the volume of the cone extending from $x = 0$ to $x = 6$.
34. Use the disc method to verify that the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.
35. Use the disc method to verify that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.
36. A sphere of radius r is cut by a plane h ($h < r$) units above the equator. Find the volume of the solid (spherical segment) above the plane.
37. A cone with a base of radius r and height H is cut by a plane to parallel to and h units above the base. Find the volume of the solid (frustrum of a cone) below the plane.
38. The region bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 4$ is revolved about the x-axis. (a) Find the value of x in the interval $[0, 4]$ that divides the solid into two parts of equal volume. (b) Find the values of x in the interval $[0, 4]$ that divide the solid into three parts of equal volume.
39. In the Chapter 6 Application, we showed that a solid of revolution having the form of a regulation-size football can be generated by revolving the graph of

$$f(x) = -0.0944x^2 + 3.4, \quad -5.5 \le x \le 5.5$$

about the x-axis. (The x and y values are measured in inches.) Show that the volume of this solid is approximately 232 cubic inches.
40. A tank on a water tower is a sphere of radius 50 feet. Determine the depth of the water when the tank is filled to 21.6 percent of its total capacity.
41. A tank on the wing of a jet is formed by revolving the region bounded by the graph of $y = \frac{1}{8}x^2\sqrt{2 - x}$ and the x-axis about the x-axis (see figure), where x and y are measured in meters. Find the volume of the tank.

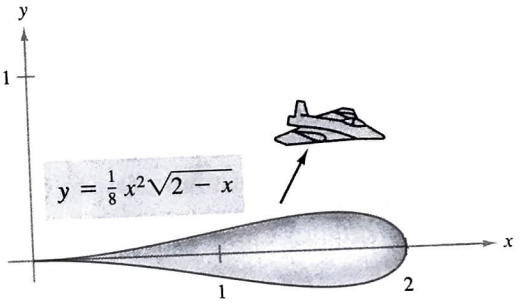


FIGURE FOR 41

In Exercises 42 and 43, use Simpson's Rule (with $n = 4$) to approximate the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the x-axis.

42. $y = \sqrt[3]{x^2 - 1}, y = 0, x = 1, x = 3$
43. $y = \sqrt[4]{x^2 + 1}, y = 0, x = 0, x = 2$

44. Match each of the following integrals with the solid whose volume it represents, and give the dimensions of each solid.

(a) right circular cylinder
(b) ellipsoid
(c) sphere
(d) right circular cone
(e) torus

___ $\pi \int_0^h \left(\frac{rx}{h}\right)^2 dx$

___ $\pi \int_0^h r^2\, dx$

___ $\pi \int_{-r}^{r} (\sqrt{r^2 - x^2})^2\, dx$

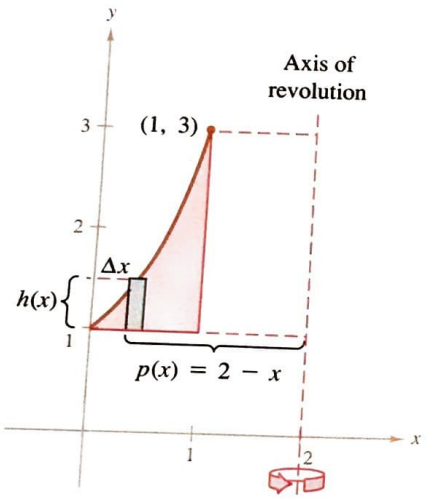
___ $\pi \int_{-b}^{b} \left(a\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$

___ $\pi \int_{-r}^{r} [(R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2]\, dx$

Note that for the shell method in Example 4 we had to solve for x in terms of y in the equation $y = 1 - (x^2/16)$. Sometimes solving for x is very difficult (or even impossible). In such cases we must use a vertical rectangle (of width Δx), thus making x the variable of integration. The position (horizontal or vertical) of the axis of revolution then determines the method to be used. This is illustrated in Example 5.

EXAMPLE 5 Shell method necessary

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^3 + x + 1$, $y = 1$, and $x = 1$ about the line $x = 2$, as shown in Figure 6.35.



$h(x) = (x^3 + x + 1) - 1$
$p(x) = 2 - x$

FIGURE 6.35

SOLUTION

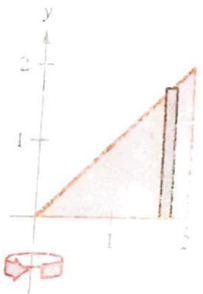
In the equation $y = x^3 + x + 1$, we cannot easily solve for x in terms of y. (See the discussion at the end of Section 4.8.) Therefore, the variable of integration must be x, and we choose a vertical representative rectangle. Since the rectangle is parallel to the axis of revolution, we use the shell method and obtain

$$\begin{aligned} V = 2\pi \int_a^b p(x)h(x)\,dx &= 2\pi \int_0^1 (2 - x)(x^3 + x + 1 - 1)\,dx \\ &= 2\pi \int_0^1 (-x^4 + 2x^3 - x^2 + 2x)\,dx \\ &= 2\pi \left[-\frac{x^5}{5} + \frac{x^4}{2} - \frac{x^3}{3} + x^2 \right]_0^1 \\ &= 2\pi \left(-\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1 \right) = \frac{29\pi}{15}. \end{aligned}$$

EXERCISES for Section 6.3

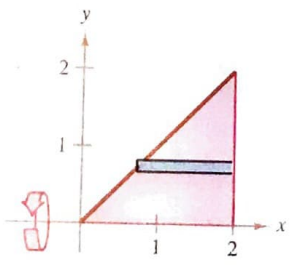
In Exercises 1–20, use the shell method to find the volume of the solid generated by revolving the given plane region about the indicated line.

1. $y = x$

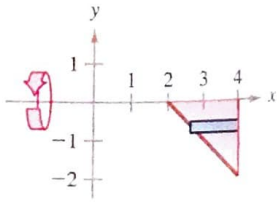


2. $y = 1 - x$

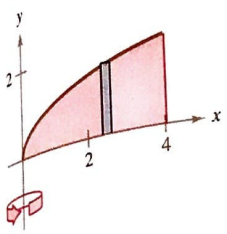
3. $y = x$



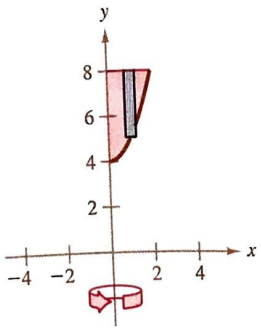
4. $y = 2 - x$



5. $y = \sqrt{x}$



6. $y = x^2 + 4$



7. $y = x^2$, $y = 0$, $x = 2$, about the y-axis
8. $y = x^2$, $y = 0$, $x = 4$, about the y-axis
9. $y = x^2$, $y = 4x - x^2$, about the y-axis
10. $y = x^2$, $y = 4x - x^2$, about the line $x = 2$
11. $y = x^2$, $y = 4x - x^2$, about the line $x = 4$
12. $y = \dfrac{1}{x}$, $x = 1$, $x = 2$, $y = 0$, about the x-axis
13. $y = 4x - x^2$, $y = 0$, about the line $x = 5$
14. $x + y^2 = 9$, $x = 0$, about the x-axis
15. $y = 4x - x^2$, $x = 0$, $y = 4$, about the y-axis
16. $y = 4 - x^2$, $y = 0$, about the y-axis
17. $y = \sqrt{x}$, $y = 0$, $x = 4$, about the line $x = 6$
18. $y = 2x$, $y = 4$, $x = 0$, about the y-axis
19. $x = y - y^2$, $x = 0$, about the line $y = 2$
20. $x = y - y^2$, $x = 0$, about the line $y = -4$

In Exercises 21–24, use the disc *or* shell method to find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the specified line.

21. $y = x^3$, $y = 0$, $x = 2$
(a) the x-axis (b) the y-axis
(c) the line $x = 4$ (d) the line $y = 8$

22. $y = \dfrac{1}{x^3}$, $y = 0$, $x = 1$, $x = 2$
(a) the x-axis (b) the y-axis
(c) the line $x = 4$

23. $x^{1/2} + y^{1/2} = a^{1/2}$, $x = 0$, $y = 0$
(a) the x-axis (b) the y-axis
(c) the line $x = a$

24. $x^{2/3} + y^{2/3} = a^{2/3}$ (hypocycloid)
(a) the x-axis (b) the y-axis

25. A solid is generated by revolving the region bounded by $y = \frac{1}{2}x^2$ and $y = 2$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-quarter of the volume is removed. Find the diameter of the hole.

26. A solid is generated by revolving the region bounded by $y = \sqrt{9 - x^2}$ and $y = 0$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-third of the volume is removed. Find the diameter of the hole.

27. Let a sphere of radius r be cut by a plane, thus forming a segment of height h. Show that the volume of this segment is $\frac{1}{3}\pi h^2(3r - h)$.

In Exercises 28 and 29, use Simpson's Rule (with $n = 4$) to approximate the volume of the solid. The solid is generated by revolving the region bounded by the graphs of the given equations about the y-axis.

28. $y = \sqrt{1 - x^3}$, $y = 0$, $x = 0$
29. $x^{4/3} + y^{4/3} = 1$

30. A pond is approximately circular with a diameter of 400 feet (see figure). Starting at the center, the depth of the water is measured every 25 feet and recorded in the accompanying table. Use Simpson's Rule to approximate the volume of water in the pond.

x	0	25	50	75	100
Depth	20	19	19	17	15

x	125	150	175	200
Depth	14	10	6	0

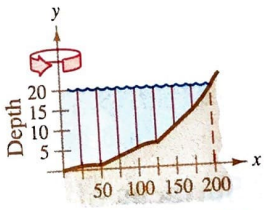


FIGURE FOR 30

31. A storage shed has a circular base of diameter 80 feet (see figure). Starting at the center, the interior height is measured every 10 feet and recorded in the accompanying table. Use Simpson's Rule to approximate the volume of the building.

x	0	10	20	30	40
Height	50	46	40	20	0

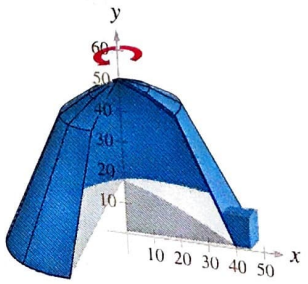


FIGURE FOR 31

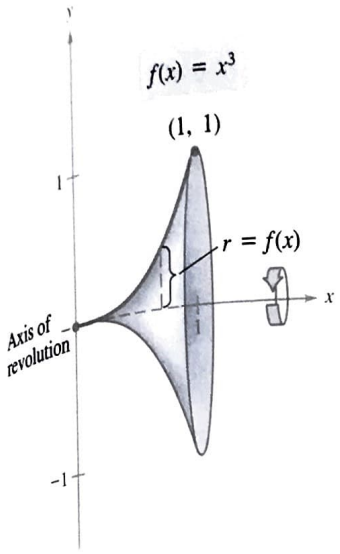


FIGURE 6.45

SOLUTION

The distance between the x-axis and the graph of f is $r(x) = f(x)$, and since $f'(x) = 3x^2$, the surface area is given by

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx = 2\pi \int_0^1 x^3\sqrt{1 + (3x^2)^2}\,dx$$

$$= \frac{2\pi}{36} \int_0^1 (36x^3)(1 + 9x^4)^{1/2}\,dx$$

$$= \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]_0^1$$

$$= \frac{\pi}{27}(10^{3/2} - 1) \approx 3.563.$$

EXAMPLE 6 The area of a surface of revolution

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis, as shown in Figure 6.46.

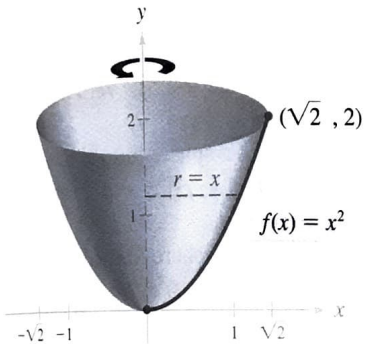


FIGURE 6.46

SOLUTION

In this case, the distance between the graph of f and the y-axis is $r(x) = x$, and since $f'(x) = 2x$, the surface area is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\,dx = 2\pi \int_0^{\sqrt{2}} x\sqrt{1 + (2x)^2}\,dx$$

$$= \frac{2\pi}{8} \int_0^{\sqrt{2}} (1 + 4x^2)^{1/2}(8x)\,dx$$

$$= \frac{2\pi}{8}\left[\frac{(1 + 4x^2)^{3/2}}{3/2}\right]_0^{\sqrt{2}}$$

$$= \frac{\pi}{6}[(1 + 8)^{3/2} - (1)^{3/2}] = \frac{13\pi}{3}.$$

EXERCISES for Section 6.4

In Exercises 1 and 2, find the distance between the given points by (a) using the distance formula and (b) determining the equation of the line through the points and using the formula for arc length.

1. (0, 0), (5, 12)
2. (1, 2), (7, 10)

In Exercises 3–8, find the arc length of the graph of the given function over the indicated interval.

	Function	*Interval*
3.	$y = \frac{2}{3}x^{3/2} + 1$	[0, 1]
4.	$y = x^{3/2} - 1$	[0, 4]
5.	$y = \frac{x^4}{8} + \frac{1}{4x^2}$	[1, 2]
6.	$y = \frac{3}{2}x^{2/3}$	[1, 8]
7.	$y = \frac{x^5}{10} + \frac{1}{6x^3}$	[1, 2]
8.	$y = 2 - \frac{2}{3}x$	[0, 3]

EXERCISES for Section 7.1

In Exercises 1 and 2, evaluate each expression.

1. (a) $25^{3/2}$ (b) $81^{1/2}$ (c) 3^{-2} (d) $27^{-1/3}$

2. (a) $64^{1/3}$ (b) 5^{-4} (c) $\left(\frac{1}{8}\right)^{1/3}$ (d) $\left(\frac{1}{4}\right)^{3}$

In Exercises 3–6, use the properties of exponents to simplify each expression.

3. (a) $(5^2)(5^3)$ (b) $(5^2)(5^{-3})$ (c) $\frac{5^3}{25^2}$ (d) $\left(\frac{1}{4}\right)^2 2^6$

4. (a) $(4^2)^3$ (b) $(6^4)^{1/2}$ (c) $[(8^{-1})(8^{2/3})]^3$ (d) $(32^{3/2})(4^2)$

5. (a) $e^2(e^4)$ (b) $(e^3)^4$ (c) $(e^3)^{-2}$ (d) $\frac{e^5}{e^3}$

6. (a) $\left(\frac{1}{e}\right)^{-2}$ (b) $\left(\frac{e^5}{e^2}\right)^{-1}$ (c) e^0 (d) $\frac{1}{e^{-3}}$

In Exercises 7–16, solve for x.

7. $3^x = 81$ **8.** $5^{x+1} = 125$

9. $\left(\frac{1}{3}\right)^{x-1} = 27$ **10.** $\left(\frac{1}{5}\right)^{2x} = 625$

11. $4^3 = (x+2)^3$ **12.** $18^2 = (5x-7)^2$

13. $x^{3/4} = 8$ **14.** $(x+3)^{4/3} = 16$

15. $e^{-2x} = e^5$ **16.** $e^x = 1$

In Exercises 17 and 18, compare the given number to the number e. Is the number less than or greater than e?

17. $\left(1 + \frac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$

18. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

In Exercises 19–28, sketch the graph of the given function.

19. $y = 3^x$ **20.** $y = 3^{x-1}$

21. $y = \left(\frac{1}{3}\right)^x$ **22.** $y = 2^{x^2}$

23. $f(x) = 3^{-x^2}$ **24.** $f(x) = 3^{|x|}$

25. $h(x) = e^{x-2}$ **26.** $g(x) = -e^{x/2}$

27. $y = e^{-x^2}$ **28.** $y = e^{-x/2}$

29. Use a graphing utility to graph $f(x) = e^x$ and the given function on the same viewing rectangle. How are the two graphs related?

(a) $g(x) = f(x-2) = e^{x-2}$

(b) $h(x) = -\frac{1}{2}f(x) = -\frac{1}{2}e^x$

(c) $q(x) = f(-x) + 3 = e^{-x} + 3$

30. Use a graphing utility to graph the function. Use the graph to determine any asymptotes of the function.

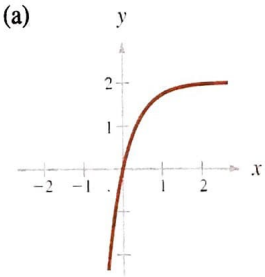
(a) $f(x) = \frac{8}{1 + e^{-0.5x}}$ (b) $g(x) = \frac{8}{1 + e^{-0.5/x}}$

In Exercises 31–34, match the equation with the correct graph. Assume that a and C are arbitrary real numbers such that $a > 0$. [The graphs are labeled (a)–(d).]

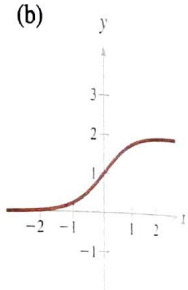
31. $y = Ce^{ax}$ **32.** $y = Ce^{-ax}$

33. $y = C(1 - e^{-ax})$ **34.** $y = \frac{C}{1 + e^{-ax}}$

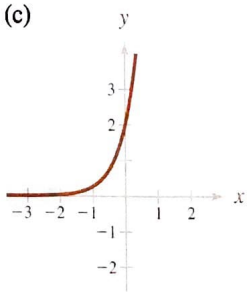
(a)



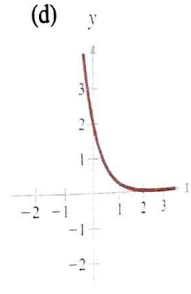
(b)



(c)



(d)



In Exercises 35 and 36, find the amount of an investment of P dollars invested at r percent for t years if the interest is compounded (a) annually, (b) semiannually, (c) monthly, (d) daily, and (e) continuously.

35. $P = \$1000$, $r = 10\%$, $t = 10$ years

36. $P = \$2500$, $r = 12\%$, $t = 20$ years

In Exercises 37 and 38, find the investment that would be required at r percent compounded continuously to yield an amount of $100,000 in (a) 1 year, (b) 10 years, (c) 20 years, and (d) 50 years.

37. $r = 12\%$ **38.** $r = 9\%$

35. Find an equation of the line normal to the graph of $y = e^{-x}$ at (0, 1).
36. Find the point on the graph of $y = e^{-x}$ where the normal line to the curve will pass through the origin.
37. Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.
38. Find, to three decimal places, the value of x such that $e^{-x} = x$. [Use Newton's Method.]

In Exercises 39 and 40, use a graphing utility to graph the function. Then graph

$$P_1(x) = f(0) + f'(0)(x - 0)$$

and

$$P_2(x) = f(0) + f'(0)(x - 0) + \frac{1}{2}f''(0)(x - 0)^2$$

in the same viewing rectangle. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 0$.

39. $f(x) = e^{x/2}$

40. $f(x) = e^{-x^2/2}$

41. Consider the function

$$f(x) = \frac{2}{1 + e^{1/x}}.$$

(a) Use a graphing utility to graph f.
(b) Write a short paragraph explaining why the graph has a horizontal asymptote $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

42. The value V of an item t years after it is purchased is

$$V = 15{,}000e^{-0.6286t}, \quad 0 \le t \le 10.$$

(a) Use a graphing utility to graph the function.
(b) Find the rate of change of V with respect to t when $t = 1$ and $t = 5$.
(c) Sketch the tangent line to the graph of the function when $t = 1$ and $t = 5$.

In Exercises 43–62, evaluate the integral.

43. $\int_0^1 e^{-2x}\,dx$

44. $\int_1^2 e^{1-x}\,dx$

45. $\int_0^2 (x^2 - 1)e^{x^3-3x+1}\,dx$

46. $\int x^2e^{x^3}\,dx$

47. $\int \frac{e^{-x}}{(1 + e^{-x})^2}\,dx$

48. $\int \frac{e^{2x}}{(1 + e^{2x})^2}\,dx$

49. $\int xe^{ax^2}\,dx$

50. $\int_0^{\sqrt{2}} xe^{-(x^2/2)}\,dx$

51. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$

52. $\int (e^x - e^{-x})^2\,dx$

53. $\int e^{-x}(1 + e^{-x})^2\,dx$

54. $\int e^{2x}(1 - 3e^{2x})^{-2}\,dx$

55. $\int e^x\sqrt{1 - e^x}\,dx$

56. $\int e^x(e^x - e^{-x})\,dx$

57. $\int \frac{e^x + e^{-x}}{\sqrt{e^x - e^{-x}}}\,dx$

58. $\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

59. $\int \frac{5 - e^x}{e^{2x}}\,dx$

60. $\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx$

61. $\int_{-2}^{0} (3^3 - 5^2)\,dx$

62. $\int (3 - x)e^{(3-x)^2}\,dx$

In Exercises 63 and 64, find a function f that satisfies the given conditions.

63. $f''(x) = \frac{1}{2}(e^x + e^{-x})$, $f(0) = 1, f'(0) = 0$

64. $f''(x) = x + e^{-2x}$, $f(0) = 3, f'(0) = -\frac{1}{2}$

In Exercises 65–68, find the area of the region bounded by the graphs of the given equations.

65. $y = e^x, y = 0, x = 0, x = 5$
66. $y = e^{-x}, y = 0, x = a, x = b$
67. $y = xe^{-(x^2/2)}, y = 0, x = 0, x = \sqrt{2}$
68. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

In Exercises 69 and 70, find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the x-axis.

69. $y = e^x, y = 0, x = 0, x = 1$
70. $y = e^{-x/2}, y = 0, x = 0, x = 4$

71. Given $e^x \ge 1$ for $x \ge 0$, it follows that

$$\int_0^x e^t\,dt \ge \int_0^x 1\,dt.$$

Perform this integration to derive the inequality $e^x \ge 1 + x$ for $x \ge 0$.

72. Integrate each term of the following inequalities in a manner similar to that of Exercise 71 to obtain each succeeding inequality for $x \ge 0$. Then evaluate both sides of each inequality when $x = 1$.
(a) $e^x \ge 1 + x$
(b) $e^x \ge 1 + x + \frac{x^2}{2}$
(c) $e^x \ge 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$
(d) $e^x \ge 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

Thus, in both cases, the slopes are reciprocals, as shown in Figure 7.18.

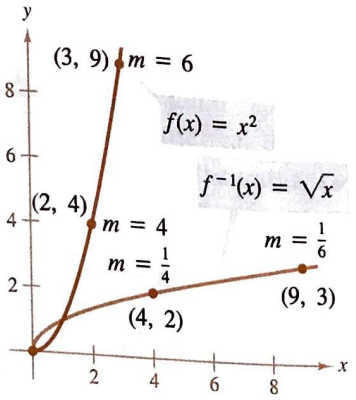


FIGURE 7.18

REMARK In Example 5, note that f^{-1} is not differentiable at (0, 0). This is consistent with Theorem 6.10 because the derivative of f at (0, 0) is zero.

EXERCISES for Section 7.3

In Exercises 1–8, (a) show that f and g are inverse functions by showing that $f(g(x)) = x$ and $g(f(x)) = x$, and (b) graph f and g on the same set of coordinate axes.

1. $f(x) = x^3$ $\quad g(x) = \sqrt[3]{x}$
2. $f(x) = x^{-1}$ $\quad g(x) = x^{-1}$
3. $f(x) = 5x + 1$ $\quad g(x) = \frac{1}{5}(x - 1)$
4. $f(x) = 3 - 4x$ $\quad g(x) = \frac{1}{4}(3 - x)$
5. $f(x) = \sqrt{x - 4}$ $\quad g(x) = x^2 + 4, x \geq 0$
6. $f(x) = 9 - x^2, x \geq 0$ $\quad g(x) = \sqrt{9 - x}$
7. $f(x) = 1 - x^3$ $\quad g(x) = \sqrt[3]{1 - x}$
8. $f(x) = x^{-2}, x > 0$ $\quad g(x) = x^{-1/2}, x > 0$

In Exercises 9–22, find the inverse of f. Then graph both f and f^{-1}.

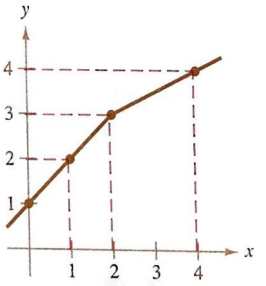
9. $f(x) = 2x - 3$
10. $f(x) = 3x$
11. $f(x) = x^5$
12. $f(x) = x^3 + 1$
13. $f(x) = \sqrt{x}$
14. $f(x) = x^2, x \geq 0$
15. $f(x) = \sqrt{4 - x^2}, 0 \leq x$
16. $f(x) = \sqrt{x^2 - 4}, x \geq 2$
17. $f(x) = \sqrt[3]{x - 1}$
18. $f(x) = 3\sqrt[5]{2x - 1}$
19. $f(x) = x^{2/3}, x \geq 0$
20. $f(x) = x^{3/5}$
21. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}}$
22. $f(x) = \dfrac{x + 2}{x}$

In Exercises 23 and 24, find the inverse function of f over the specified interval. Use a graphing utility to graph f and f^{-1} in the same viewing rectangle. Observe that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

Function	*Interval*
23. $f(x) = \dfrac{x}{x^2 - 4}$	$-2 < x < 2$
24. $f(x) = 2 - \dfrac{3}{x^2}$	$0 < x < 10$

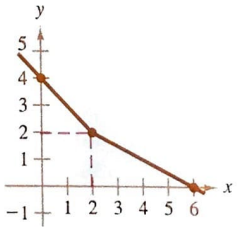
In Exercises 25 and 26, use the graph of the function f to complete the table and sketch the graph of f^{-1}.

25.



x	1	2	3	4
$f^{-1}(x)$				

26.



x	0	2	4
$f^{-1}(x)$			

In Exercises 27–30, use the functions

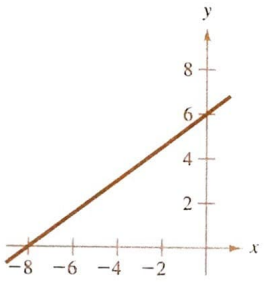
$$f(x) = \frac{1}{8}x - 3 \quad \text{and} \quad g(x) = x^3$$

to find the indicated value.

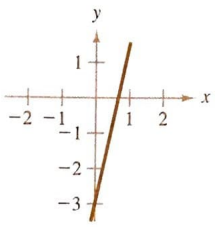
27. $(f^{-1} \circ g^{-1})(1)$
28. $(g^{-1} \circ f^{-1})(-3)$
29. $(f^{-1} \circ f^{-1})(6)$
30. $(g^{-1} \circ g^{-1})(-4)$

In Exercises 31–36, use the horizontal line test to determine whether the function is one-to-one on its entire domain and therefore has an inverse.

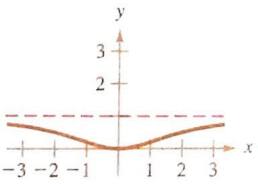
31. $f(x) = \frac{3}{4}x + 6$



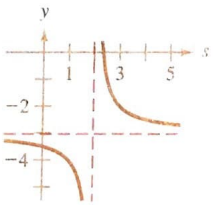
32. $f(x) = 5x - 3$



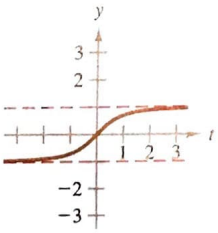
33. $f(x) = \frac{x^2}{x^2 + 4}$



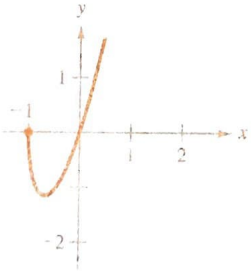
34. $h(s) = \frac{1}{s - 2} - 3$



35. $g(t) = \frac{t}{\sqrt{t^2 + 1}}$



36. $f(x) = 3x\sqrt{x + 1}$



In Exercises 37–40, use the derivative to determine whether the given function is strictly monotonic on its entire domain and therefore has an inverse.

37. $f(x) = (x + a)^3 + b$
38. $f(x) = \frac{x^4}{4} - 2x^2$
39. $f(x) = x^3 - 6x^2 + 12x$
40. $f(x) = 2 - x - x^3$

In Exercises 41–44, use a graphing utility to graph the function. From the graph, determine whether the function is one-to-one on its entire domain.

41. $h(s) = \frac{1}{s - 2} - 3$
42. $g(t) = \frac{1}{\sqrt{t^2 + 1}}$
43. $h(x) = |x + 4| - |x - 4|$
44. $g(x) = (x + 5)^3$

In Exercises 45–48, show that f is strictly monotonic on the given domain and therefore has an inverse on that domain.

	Function	*Domain*
45.	$f(x) = (x - 4)^2$	$[4, \infty)$
46.	$f(x) = \lvert x + 2 \rvert$	$[-2, \infty)$
47.	$f(x) = \frac{4}{x^2}$	$(0, \infty)$
48.	$f(x) = x^3 - x$	$[1, \infty)$

In Exercises 49–52, show that the slopes of the graphs of f and f^{-1} are reciprocals at the given points.

	Functions	*Point*
49.	$f(x) = x^3$	$\left(\frac{1}{2}, \frac{1}{8}\right)$
	$f^{-1}(x) = \sqrt[3]{x}$	$\left(\frac{1}{8}, \frac{1}{2}\right)$
50.	$f(x) = 3 - 4x$	$(1, -1)$
	$f^{-1}(x) = \frac{3 - x}{4}$	$(-1, 1)$
51.	$f(x) = \sqrt{x - 4}$	$(5, 1)$
	$f^{-1}(x) = x^2 + 4$	$(1, 5)$
52.	$f(x) = \frac{1}{1 + x^2}$	$\left(1, \frac{1}{2}\right)$
	$f^{-1}(x) = \sqrt{\frac{1 - x}{x}}$	$\left(\frac{1}{2}, 1\right)$

In Exercises 53 and 54, the derivative of the function has the same sign for all x in its domain, but the function is not strictly monotonic. Explain why.

53. $f(x) = \frac{1}{x}$
54. $f(x) = \frac{x}{x^2 - 4}$

55. Prove that if a function has an inverse, then the inverse is unique.
56. Prove that if f has an inverse, then $(f^{-1})^{-1} = f$.
57. Prove that a function has an inverse if and only if it is one-to-one.
58. Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

Most calculators have both natural and common logarithmic keys (often denoted by $\ln x$ and $\log x$). Suppose, however, that you are asked to evaluate a logarithm to a base other than e or 10. In such cases, you can use the following change-of-base formula.

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{Change-of-base formula}$$

With this formula, we can use a calculator to evaluate expressions such as $\log_2 14$. That is,

$$\log_2 14 = \frac{\ln 14}{\ln 2} \approx \frac{2.63906}{0.693147} \approx 3.8074.$$

EXERCISES for Section 7.4

In Exercises 1–6, write the logarithmic equation as an exponential equation and vice versa.

1. (a) $2^3 = 8$ (b) $3^{-1} = \frac{1}{3}$

2. (a) $27^{2/3} = 9$ (b) $16^{3/4} = 8$

3. (a) $\log_{10} 0.01 = -2$ (b) $\log_{0.5} 8 = -3$

4. (a) $e^0 = 1$ (b) $e^2 = 7.389\ldots$

5. (a) $\ln 2 = 0.6931\ldots$ (b) $\ln 8.4 = 2.128\ldots$

6. (a) $\ln 0.5 = -0.6931\ldots$ (b) $49^{1/2} = 7$

In Exercises 7–14, solve for x (or b).

7. (a) $\log_{10} 1000 = x$ (b) $\log_{10} 0.1 = x$

8. (a) $\log_4 \frac{1}{64} = x$ (b) $\log_5 25 = x$

9. (a) $\log_3 x = -1$ (b) $\log_2 x = -4$

10. (a) $\log_b 27 = 3$ (b) $\log_b 125 = 3$

11. (a) $\log_{27} x = -\frac{2}{3}$ (b) $\ln e^x = 3$

12. (a) $e^{\ln x} = 4$ (b) $\ln x = 2$

13. (a) $x^2 - x = \log_5 25$
(b) $3x + 5 = \log_2 64$

14. (a) $\log_3 x + \log_3 (x - 2) = 1$
(b) $\log_{10} (x + 3) - \log_{10} x = 1$

In Exercises 15–20, sketch the graph of the function.

15. $f(x) = 3 \ln x$

16. $f(x) = -2 \ln x$

17. $f(x) = \ln 2x$

18. $f(x) = \ln |x|$

19. $f(x) = \ln (x - 1)$

20. $f(x) = 2 + \ln x$

In Exercises 21–24, show that the given functions are inverses of each other by sketching their graphs on the same coordinate axes.

21. $f(x) = e^{2x}$, $g(x) = \ln \sqrt{x}$

22. $f(x) = e^x - 1$, $g(x) = \ln (x + 1)$

23. $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$

24. $f(x) = e^{x/3}$, $g(x) = \ln x^3$

In Exercises 25–30, apply the inverse properties of $\ln x$ and e^x to simplify the given expression.

25. $\ln e^{x^2}$

26. $\ln e^{2x-1}$

27. $e^{\ln (5x+2)}$

28. $-1 + \ln e^{2x}$

29. $e^{\ln \sqrt{x}}$

30. $-8 + e^{\ln x^3}$

In Exercises 31 and 32, use the properties of logarithms and the fact that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$ to approximate the given logarithm.

31. (a) $\ln 6$ (b) $\ln \frac{2}{3}$
(c) $\ln 81$ (d) $\ln \sqrt{3}$

32. (a) $\ln 0.25$ (b) $\ln 24$
(c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

In Exercises 33–42, use the properties of logarithms to write each as a sum, difference, or multiple of logarithms.

33. $\ln \frac{2}{3}$

34. $\ln (xyz)$

35. $\ln \frac{xy}{z}$

36. $\ln \sqrt{a - 1}$

37. $\ln \sqrt{2^3}$

38. $\ln \frac{1}{5}$

39. $\ln \left(\frac{x^2 - 1}{x^3}\right)^3$

40. $\ln 3e^2$

41. $\ln z(z - 1)^2$

42. $\ln \frac{1}{e}$

In Exercises 43–48, write each expression as a logarithm of a single quantity.

43. $\ln(x-2) - \ln(x+2)$
44. $3\ln x + 2\ln y - 4\ln z$
45. $\frac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)]$
46. $2[\ln x - \ln(x+1) - \ln(x-1)]$
47. $2\ln 3 - \frac{1}{2}\ln(x^2+1)$
48. $\frac{3}{2}[\ln(x^2+1) - \ln(x+1) - \ln(x-1)]$

In Exercises 49–60, solve for x or t.

49. $e^{\ln x} = 4$
50. $e^{\ln x^2} - 9 = 0$
51. $\ln x = 0$
52. $2\ln x = 4$
53. $e^{x+1} = 4$
54. $e^{-0.5x} = 0.075$
55. $500e^{-0.11t} = 600$
56. $e^{-0.0174t} = 0.5$
57. $5^{2x} = 15$
58. $2^{1-x} = 6$
59. $500(1.07)^t = 1000$
60. $1000\left(1 + \frac{0.07}{12}\right)^{12t} = 3000$

61. A deposit of \$1000 is made into a fund with an annual interest rate of 11%. Find the time for the investment to double if the interest is compounded
(a) annually. (b) monthly.
(c) daily. (d) continuously.

62. A deposit of \$1000 is made into a fund with an annual interest rate of $10\frac{1}{2}$%. Find the time for the investment to triple if the interest is compounded
(a) annually. (b) monthly.
(c) daily. (d) continuously.

63. Complete the following table for the time t necessary for P dollars to triple if interest is compounded continuously at the rate r.

r	2%	4%	6%	8%	10%	12%
t						

64. The demand function for a certain product is given by

$$p = 500 - 0.5e^{0.004x}.$$

Find the quantity x demanded for a price of (a) $p =$ \$350 and (b) $p =$ \$300.

65. Use a calculator to demonstrate that

$$\frac{\ln x}{\ln y} \neq \ln\frac{x}{y} = \ln x - \ln y$$

by completing the following table.

x	y	$\frac{\ln x}{\ln y}$	$\ln\frac{x}{y}$	$\ln x - \ln y$
1	2			
3	4			
10	5			
4	0.5			

66. There are 25 prime numbers less than 100. The **Prime Number Theorem** states that if $p(x)$ is the number of primes less than x, then the ratio of $p(x)$ to $x/\ln x$ approaches 1 as x approaches infinity. Compute $x/\ln x$ for $x = 1000$, $x = 1{,}000{,}000$, and $x = 1{,}000{,}000{,}000$. Then compute the ratio of $p(x)$ to $x/\ln x$ given that $p(1000) = 168$, $p(10^6) = 78{,}498$, and $p(10^9) = 50{,}847{,}478$.

In Exercises 67 and 68, show that $f = g$ by using a computer or graphics calculator to sketch the graph of f and g on the same coordinate axes. (Assume $x > 0$.)

67. $f(x) = \ln\frac{x^2}{4}$
$g(x) = 2\ln x - \ln 4$
68. $f(x) = \ln\sqrt{x(x^2+1)}$
$g(x) = \frac{1}{2}[\ln x + \ln(x^2+1)]$

In Exercises 69 and 70, evaluate the logarithm using the change of base formula. Do each problem twice. The first time use common logarithms, and the second time use natural logarithms. Round your answer to three decimal places.

69. (a) $\log_3 7$ (b) $\log_7 4$
(c) $\log_{1/2} 10$ (d) $\log_4 0.55$
70. (a) $\log_9 0.4$ (b) $\log_2 0.125$
(c) $\log_{15} 1250$ (d) $\log_{1/3} 0.015$

71. Prove that

$$\ln\frac{x}{y} = \ln x - \ln y.$$

72. Prove that $\ln x^y = y\ln x$.

EXAMPLE 10 Comparing variables and constants

(a) $\dfrac{d}{dx}[e^e] = 0$ — Constant Rule

(b) $\dfrac{d}{dx}[e^x] = e^x$ — Exponential Rule

(c) $\dfrac{d}{dx}[x^e] = ex^{e-1}$ — Power Rule

(d) $y = x^x$ — Logarithmic differentiation

$$\ln y = x \ln x$$

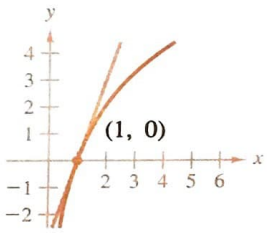
$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$

$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

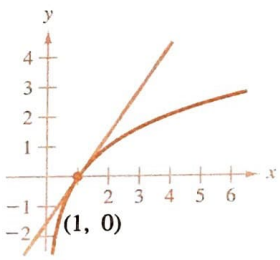
EXERCISES for Section 7.5

In Exercises 1–4, find the slope of the tangent line to the given logarithmic function at the point (1, 0).

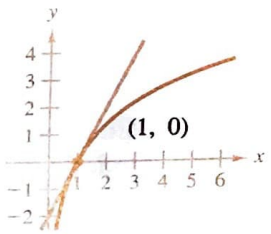
1. $y = \ln x^3$



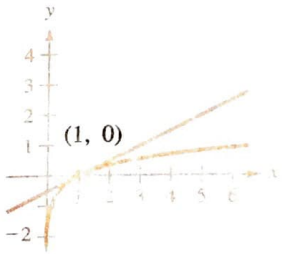
2. $y = \ln x^{3/2}$



3. $y = \ln x^2$



4. $y = \ln x^{1/2}$



In Exercises 5–38, find dy/dx.

5. $y = \ln x^2$

6. $y = \ln (x^2 + 3)$

7. $y = \ln \sqrt{x^4 - 4x}$

8. $y = \ln (1 - x)^{3/2}$

9. $y = (\ln x)^4$

10. $y = x \ln x$

11. $y = \ln (x\sqrt{x^2 - 1})$

12. $y = \ln \left(\dfrac{x}{x + 1}\right)$

13. $y = \ln \left(\dfrac{x}{x^2 + 1}\right)$

14. $y = \dfrac{\ln x}{x}$

15. $y = \dfrac{\ln x}{x^2}$

16. $y = \ln (\ln x)$

17. $y = \ln (\ln x^2)$

18. $y = \ln \sqrt{\dfrac{x - 1}{x + 1}}$

19. $y = \ln \sqrt{\dfrac{x + 1}{x - 1}}$

20. $y = \ln \sqrt{x^2 - 4}$

21. $y = \ln \left(\dfrac{\sqrt{4 + x^2}}{x}\right)$

22. $y = \ln (x + \sqrt{4 + x^2})$

23. $y = \dfrac{-\sqrt{x^2 + 1}}{x} + \ln (x + \sqrt{x^2 + 1})$

24. $y = \dfrac{-\sqrt{x^2 + 4}}{2x^2} - \dfrac{1}{4} \ln \left(\dfrac{2 + \sqrt{x^2 + 4}}{x}\right)$

25. $y = 4^x$

26. $y = 2^{-x}$

27. $y = 5^{x-2}$

28. $y = x(7^{-3x})$

29. $y = x^2 2^x$

30. $y = 2^{x^2} 3^{-x}$

31. $y = \log_3 x$

32. $y = \log_{10} 2x$

33. $y = \log_2 \left(\dfrac{x^2}{x - 1}\right)$

34. $y = \log_3 \left(\dfrac{x\sqrt{x - 1}}{2}\right)$

35. $y = \log_5 \sqrt{x^2 - 1}$

36. $y = \log_{10} \left(\dfrac{x^2 - 1}{x}\right)$

37. $y = \ln |x^2 - 1|$

38. $y = \ln \left|\dfrac{x + 5}{x}\right|$

In Exercises 39–48, find dy/dx using logarithmic differentiation.

39. $y = x\sqrt{x^2 - 1}$

40. $y = \sqrt{(x - 1)(x - 2)(x - 3)}$

41. $y = \dfrac{x^2\sqrt{3x - 2}}{(x - 1)^2}$

42. $y = \sqrt[3]{\dfrac{x^2 + 1}{x^2 - 1}}$

43. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}$

44. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

45. $y = x^{2/x}$

46. $y = x^{x-1}$

47. $y = (x - 2)^{x+1}$

48. $y = (1 + x)^{1/x}$

In Exercises 49 and 50, show that the given function is a solution to the differential equation.

	Function	*Differential equation*
49.	$y = 2 \ln x + 3$	$x(y'') + y' = 0$
50.	$y = x \ln x - 4x$	$x + y - xy' = 0$

In Exercises 51 and 52, find dy/dx by using implicit differentiation.

51. $x^2 - 3 \ln y + y^2 = 10$
52. $\ln xy + 5x = 30$

In Exercises 53 and 54, find an equation of the tangent line to the graph of the equation at the given point.

	Equation	*Point*
53.	$y = 3x^2 - \ln x$	$(1, 3)$
54.	$x^2 + \ln (x + 1) + y^2 = 4$	$(0, 2)$

In Exercises 55–60, find any relative extrema and inflection points, and sketch the graph of the function.

55. $y = \dfrac{x^2}{2} - \ln x$

56. $y = x - \ln x$

57. $y = x (\ln x)$

58. $y = \dfrac{\ln x}{x}$

59. $y = \dfrac{x}{\ln x}$

60. $y = x^2 (\ln x)$

In Exercises 61 and 62, use Newton's Method to approximate, to three decimal places, the x-coordinate of the point of intersection of the graphs of the two equations.

61. $y = \ln x$, $y = -x$

62. $y = \ln x$, $y = 3 - x$

63. Let $L(x) = \displaystyle\int_1^x (1/t)\, dt$ for all $x > 0$.
   (a) Find $L(1)$.
   (b) Find $L'(x)$ and $L'(1)$.
   (c) Use the Trapezoidal Rule to approximate the value of x (to three decimal places) for which $L(x) = 1$.
   (d) Show that L is an increasing function on $(0, \infty)$.
   (e) Prove that $L(x_1x_2) = L(x_1) + L(x_2)$ for $x_1 > 0$ and $x_2 > 0$.

64. Show that

$$f(x) = \frac{\ln x^n}{x}$$

is a decreasing function for $x > e$ and $n > 0$.

65. A person walking along a dock drags a boat by a 10-foot rope. The boat travels along a path known as a *tractrix* (see figure). The equation of the path is

$$y = 10 \ln \left(\frac{10 + \sqrt{100 - x^2}}{x} \right) - \sqrt{100 - x^2}.$$

What is the slope of this path at the following x-values?
(a) $x = 10$ (b) $x = 5$

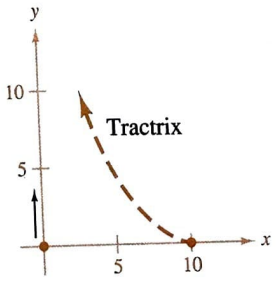


FIGURE FOR 65

7.6 Logarithmic Functions and Integration

The Log Rule for integration

The differentiation rules

$$\frac{d}{dx} [\ln |x|] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx} [\ln |u|] = \frac{u'}{u}$$

allow us to patch up the hole in our General Power Rule for integration. Recall from Section 5.5 that

$$\int u^n \, du = \frac{u^{n+1}}{n + 1} + C$$

provided $n \neq -1$. Having the differentiation formulas for logarithmic functions, we are now in a position to evaluate $\int u^n \, du$ for $n = -1$, as stated in the following theorem.

EXERCISES for Section 7.6

In Exercises 1–36, evaluate each integral.

1. $\int \frac{1}{x+1}\,dx$
2. $\int \frac{1}{x-5}\,dx$
3. $\int \frac{1}{3-2x}\,dx$
4. $\int \frac{1}{6x+1}\,dx$
5. $\int \frac{x}{x^2+1}\,dx$
6. $\int \frac{x^2}{3-x^3}\,dx$
7. $\int \frac{x^2-4}{x}\,dx$
8. $\int \frac{x+5}{x}\,dx$
9. $\int_1^e \frac{\ln x}{2x}\,dx$
10. $\int_e^{e^2} \frac{1}{x\,(\ln x)}\,dx$
11. $\int_1^e \frac{(1+\ln x)^2}{x}\,dx$
12. $\int_0^1 \frac{x-1}{x+1}\,dx$
13. $\int_0^2 \frac{x^2-2}{x+1}\,dx$
14. $\int \frac{1}{(x+1)^2}\,dx$
15. $\int \frac{1}{\sqrt{x+1}}\,dx$
16. $\int \frac{x+3}{x^2+6x+7}\,dx$
17. $\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$
18. $\int \frac{(\ln x)^2}{x}\,dx$
19. $\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx$
20. $\int \frac{1}{x\ln(x^2)}\,dx$
21. $\int \frac{1}{1+\sqrt{x}}\,dx$
22. $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$
23. $\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$
24. $\int_0^2 \frac{1}{1+\sqrt{2x}}\,dx$
25. $\int \frac{\sqrt{x}}{1-x\sqrt{x}}\,dx$
26. $\int \frac{2x}{(x-1)^2}\,dx$
27. $\int \frac{x(x-2)}{(x-1)^3}\,dx$
28. $\int \frac{x\sqrt{x}}{1+x^2\sqrt{x}}\,dx$
29. $\int 3^x\,dx$
30. $\int 4^{-x}\,dx$
31. $\int_{-1}^2 2^x\,dx$
32. $\int 2^3\,dx$
33. $\int x5^{x^2}\,dx$
34. $\int (3-x)7^{(3-x)^2}\,dx$
35. $\int \frac{e^{-x}}{1+e^{-x}}\,dx$
36. $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

In Exercises 37 and 38, find the area of the indicated region.

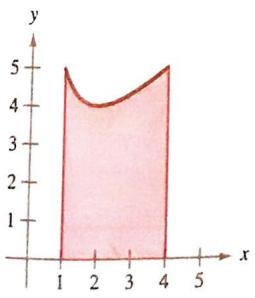


FIGURE FOR 37

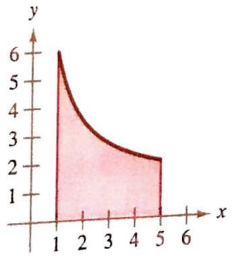


FIGURE FOR 38

37. $y = \frac{x^2+4}{x}$

38. $y = \frac{x+5}{x}$

In Exercises 39–42, find the volume of the solid generated by revolving the region bounded by the graphs of the given equations about the indicated axis.

39. $y = \frac{1}{\sqrt{x+1}}$, $y = 0$, $x = 0$, $x = 3$, about the x-axis
40. $y = \frac{1}{x(x+1)}$, $y = 0$, $x = 1$, $x = 3$, about the y-axis
41. $xy = 1$, $y = 0$, $x = 1$, $x = 4$, about the line $y = 4$
42. $y = \frac{1}{1+\sqrt{x-2}}$, $y = 0$, $x = 2$, $x = 6$, about the y-axis
43. Find the area of the region bounded by the graphs of $y = 3^x$, $y = 0$, $x = 0$, and $x = 3$.
44. Find the centroid of the region bounded by the graphs of $y = 1/x$, $y = 0$, $x = 1$, and $x = 4$.

In Exercises 45 and 46, find the work done by the gas for the given volumes and pressures. Assume the pressure is inversely proportional to the volume. (Use Example 9 as a model.)

45. A quantity of gas with an initial volume of 2 cubic feet and an initial pressure of 1000 pounds per square foot expands to a volume of 3 cubic feet.
46. A quantity of gas with an initial volume of 1 cubic foot and an initial pressure of 2000 pounds per square foot expands to a volume of 4 cubic feet.
47. A population of bacteria is changing at the rate of

$$\frac{dP}{dt} = \frac{3000}{1+0.25t}$$

where t is the time in days. Assuming that the initial population (when $t = 0$) is 1000, write an equation that gives the population at any time t, and then find the population when $t = 3$ days.

48. The demand equation for a product is given by

$$p = \frac{90{,}000}{400+3x}.$$

Find the *average* price p on the interval $40 \le x \le 50$.

49. Find the time required for an object to cool from 300° to 250° if that time is given by

$$t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T-100}\,dT.$$

SOLUTION

From Newton's Law of Cooling, we know that the rate of change in y is proportional to the difference between y and 60. This gives us

$$\frac{dy}{dt} = k(y - 60), \quad 80 \le y \le 100$$

$$\left(\frac{1}{y - 60}\right)\frac{dy}{dt} = k$$

$$\int \frac{1}{y - 60}\, dy = \int k\, dt \qquad \text{Separation of variables}$$

$$\ln |y - 60| = kt + C.$$

Since $y > 60$, we can delete the absolute value signs. Now, using $y = 100$ when $t = 0$, we have $C = \ln 40$, which implies that

$$kt = \ln (y - 60) - \ln 40 = \ln \left(\frac{y - 60}{40}\right).$$

Furthermore, since $y = 90$ when $t = 10$, we have $k = \frac{1}{10} \ln \frac{3}{4}$, and thus

$$t = \left(\frac{10}{\ln (3/4)}\right) \ln \left(\frac{y - 60}{40}\right).$$

Finally, when $y = 80$,

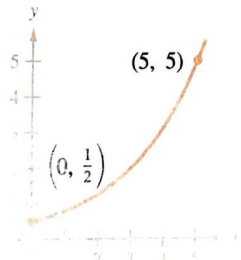
$$t = \frac{10 \ln (1/2)}{\ln (3/4)} \approx 24.09 \text{ min}.$$

Therefore, it will require approximately 14.09 *more* minutes for the object to cool to a temperature of 80°. □

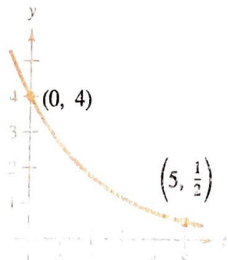
EXERCISES for Section 7.7

In Exercises 1–4, find the exponential function $y = Ce^{kt}$ that passes through the two given points.

1.



2.



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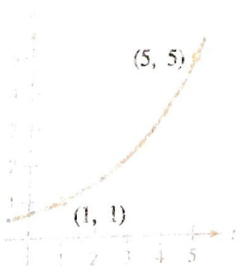

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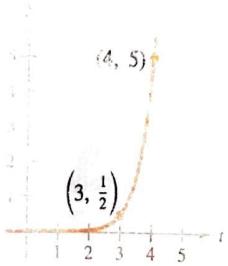


In Exercises 5–10, complete the table for a savings account in which interest is compounded continuously.

	Initial investment	*Annual rate*	*Time to double*	*Amount after 10 yrs*	*Amount after 25 yrs*
5.	\$1,000	12%			
6.	\$20,000	$10\frac{1}{2}$%			
7.	\$750		$7\frac{3}{4}$ years		
8.	\$10,000		5 years		
9.	\$500			\$1,292.85	
10.	\$2,000				\$6,008.33

11. The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units N produced per day after a new employee has worked t days is given by

$$N = 30(1 - e^{kt})$$

After 20 days on the job, a particular worker produced 19 units.
(a) Find the learning curve for this worker.
(b) How many days should pass before this worker is producing 25 units per day?

12. If in Exercise 11 the management requires a new employee to produce at least 20 units per day after 30 days on the job, find
(a) the learning curve that describes this minimum requirement,
(b) the number of days before a minimal achiever is producing 25 units per day.

13. The sales S (in thousands of units) of a new product after it is on the market t years is given by

$$S = Ce^{k/t}.$$

(a) Find S as a function of t if 5000 units have been sold after 1 year and the saturation point for the market is 30,000 (that is, $\lim_{t\to\infty} S = 30$).
(b) How many units have been sold after 5 years?
(c) Sketch a graph of this sales function.

14. The number of sales S (in thousands of units) of a new product after it has been on the market t years is given by

$$S = 30(1 - e^{kt}).$$

(a) Find S as a function of t if 5000 units have been sold after 1 year.
(b) How many units will saturate this market?
(c) How many units will have been sold after 5 years?
(d) Sketch a graph of this sales function.

15. A certain type of bacteria increases continuously at a rate proportional to the number present. If there are 100 present at a given time and 300 present 5 hours later, how many will there be 10 hours after the initial given time?

16. Given the conditions of Exercise 15, how long would it take for the number of bacteria to double?

17. In 1970 the population of a town was 2500, and in 1980 it was 3350. Assuming the population increases continuously at a constant rate proportional to the existing population, estimate the population in the year 2000.

18. Given the conditions of Exercise 17, how many years would be necessary for the population to double?

In Exercises 19–24, complete the table for the given radioactive isotope.

	Isotope	Half-life (in yrs)	Initial quantity	Amount after 1,000 yrs	Amount after 10,000 yrs
19.	Ra^{226}	1,620	10 grams		
20.	Ra^{226}	1,620		1.5 grams	
21.	C^{14}	5,730			2 grams
22.	C^{14}	5,730	3 grams		
23.	Pu^{230}	24,360		2.1 grams	
24.	Pu^{230}	24,360			0.4 gram

25. Radioactive radium has a half-life of approximately 1620 years. What percentage of a given amount remains after 100 years?

26. If radioactive material decays continuously at a rate proportional to the amount present, find the half-life of the material if after 1 year 99.57 percent of an initial amount still remains.

27. C^{14} dating assumes that the carbon dioxide on earth today has the same radioactive content as it did centuries ago. If this is true, then the amount of C^{14} absorbed by a tree that grew several centuries ago should be the same as the amount of C^{14} absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? [The half-life of C^{14} is 5730 years.]

28. On the Richter Scale, the magnitude R of an earthquake of intensity I is given by

$$R = \frac{\ln I - \ln I_0}{\ln 10}$$

where I_0 is the minimum intensity used for comparison. Assume $I_0 = 1$.
(a) Find the intensity of the 1906 San Francisco earthquake when $R = 8.3$.
(b) Find the factor by which the intensity is increased if the Richter Scale measurement is doubled.
(c) Find dR/dI.

29. Using Newton's Law of Cooling (see Example 5), determine the reading on a thermometer 5 minutes after it is taken from a room at 72° Fahrenheit to the outdoors where the temperature is 20°, if the reading dropped to 48° after 1 minute.

30. An object in a room at 70° cools from 350° to 150° in 45 minutes. Using Newton's Law of Cooling, find the time necessary for the object to cool to 80°.

31. Using Newton's Law of Cooling, determine the outdoor temperature if a thermometer is taken from a room where the temperature is 68° to the outdoors, where after $\frac{1}{2}$ minute and 1 minute the thermometer reads 53° and 42°, respectively.

32. When an object is removed from a furnace and placed in an environment with a constant temperature of 90°, its core temperature is 1500°. One hour after it is removed, the core temperature is 1120°. Find the core temperature 5 hours after the unit is removed.

In each of the examples so far in this section, we have used L'Hôpital's Rule to find a limit that exists. L'Hôpital's Rule can also be used to conclude that a limit is infinite, and this is demonstrated in the last example.

EXAMPLE 8 An infinite limit

Evaluate

$$\lim_{x\to\infty} \frac{e^x}{x}.$$

SOLUTION

Direct substitution produces the indeterminate form ∞/∞, therefore we apply L'Hôpital's Rule to obtain

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \lim_{x\to\infty} e^x = \infty.$$

Now, since $e^x \to \infty$ as $x \to \infty$, we conclude that the limit of e^x/x as $x \to \infty$ is also infinite. ▭

As a final comment, we remind you that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms $0/0$ or ∞/∞. For instance, the following application of L'Hôpital's Rule is *incorrect*.

$$\lim_{x\to 0} \frac{e^x}{x} \overset{\text{(!)}}{=} \lim_{x\to 0} \frac{e^x}{1} = 1$$

Incorrect use of L'Hôpital's Rule

The reason this application is incorrect is that, even though the limit of the denominator is 0, the limit of the numerator is 1—which means that the hypotheses of L'Hôpital's Rule have not been satisfied.

EXERCISES for Section 7.8

In Exercises 1–4, evaluate the limit (a) using techniques from Chapters 2 and 4 and (b) using L'Hôpital's Rule.

1. $\lim_{x\to 3} \dfrac{2(x-3)}{x^2-9}$
2. $\lim_{x\to -1} \dfrac{2x^2-x-3}{x+1}$
3. $\lim_{x\to\infty} \dfrac{5x^2-3x+1}{3x^2-5}$
4. $\lim_{x\to\infty} \dfrac{2x+1}{4x^2+x}$

In Exercises 5–30, evaluate each limit, using L'Hôpital's Rule if necessary.

5. $\lim_{x\to 2} \dfrac{x^2-x-2}{x-2}$
6. $\lim_{x\to -1} \dfrac{x^2-x-2}{x+1}$
7. $\lim_{x\to 0} \dfrac{\sqrt{4-x^2}-2}{x}$
8. $\lim_{x\to 2^-} \dfrac{\sqrt{4-x^2}}{x-2}$
9. $\lim_{x\to 0} \dfrac{e^x-(1-x)}{x}$
10. $\lim_{x\to 0^+} \dfrac{e^x-(1+x)}{x^3}$
11. $\lim_{x\to 0^+} \dfrac{e^x-(1+x)}{x^n}$, $n = 1, 2, 3, \ldots$
12. $\lim_{x\to 1} \dfrac{\ln x}{x^2-1}$
13. $\lim_{x\to\infty} \dfrac{\ln x}{x}$
14. $\lim_{x\to\infty} \dfrac{e^x}{x}$
15. $\lim_{x\to\infty} \dfrac{3x^2-2x+1}{2x^2+3}$
16. $\lim_{x\to\infty} \dfrac{x-1}{x^2+2x+3}$

17. $\lim_{x\to\infty} \frac{x^2 + 2x + 3}{x - 1}$

18. $\lim_{x\to\infty} \frac{x^2}{e^x}$

19. $\lim_{x\to 0^+} x^2 \ln x$

20. $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2}\right)$

21. $\lim_{x\to 2} \left(\frac{8}{x^2 - 4} - \frac{x}{x - 2}\right)$

22. $\lim_{x\to 2} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right)$

23. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}}$

24. $\lim_{x\to 1^+} \left(\frac{3}{\ln x} - \frac{2}{x - 1}\right)$

25. $\lim_{x\to 0^+} x^{1/x}$

26. $\lim_{x\to 0^+} (e^x + x)^{1/x}$

27. $\lim_{x\to\infty} x^{1/x}$

28. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

29. $\lim_{x\to\infty} (1 + x)^{1/x}$

30. $\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x - 1}}$

31. Find differentiable functions f and g that satisfy the specified condition such that $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$.

(a) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 10$ (b) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 0$

(c) $\lim_{x\to 5} \frac{f(x)}{g(x)} = \infty$

(Note: There are many correct answers.)

32. Find differentiable functions f and g such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \text{ and}$$

$$\lim_{x\to\infty} [f(x) - g(x)] = 25.$$

(Note: There are many correct answers.)

In Exercises 33–38, use L'Hôpital's Rule to determine the comparative rates of increase of the functions

$f(x) = x^m$

$g(x) = e^{nx}$

$h(x) = (\ln x)^n$

where $n > 0$, $m > 0$, and $x \to \infty$. The limits obtained in these exercises suggest that $(\ln x)^n$ tends toward infinity more slowly than x^m, which in turn tends toward infinity more slowly than e^{nx}.

33. $\lim_{x\to\infty} \frac{x^2}{e^{5x}}$

34. $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$

35. $\lim_{x\to\infty} \frac{(\ln x)^3}{x}$

36. $\lim_{x\to\infty} \frac{(\ln x)^2}{x^3}$

37. $\lim_{x\to\infty} \frac{(\ln x)^n}{x^m}$, where $0 < n, m$

38. $\lim_{x\to\infty} \frac{x^m}{e^{nx}}$, where $0 < n, m$

In Exercises 39 and 40, L'Hôpital's Rule is used incorrectly. Describe the error.

39. $\lim_{x\to 0} \frac{e^{2x} - 1}{e^x} = \lim_{x\to 0} \frac{2e^{2x}}{e^x} = \lim_{x\to 0} 2e^x = 2$

40. $\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \lim_{x\to\infty} \frac{-e^{-x}}{-e^{-x}} = \lim_{x\to\infty} 1 = 1$

41. Complete the following table to show that x eventually "overpowers" $(\ln x)^4$.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\frac{(\ln x)^4}{x}$						

42. Complete the following table to show that e^x eventually "overpowers" x^5.

x	1	5	10	20	30	40	50	100
$\frac{e^x}{x^5}$								

43. Prove that if $f(x) \geq 0$, $\lim_{x\to a} f(x) = 0$, and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} f(x)^{g(x)} = 0.$$

44. Prove that if $f(x) \geq 0$, $\lim_{x\to a} f(x) = 0$, and $\lim_{x\to a} g(x) = -\infty$, then

$$\lim_{x\to a} f(x)^{g(x)} = \infty.$$

45. Use the results of Exercises 43 and 44 to evaluate the following limits.

(a) $\lim_{x\to 0} (e^x - 1)^{1/x^2}$

(b) $\lim_{x\to 0} (e^x - 1)^{-1/x^2}$

46. The Gamma Function $\Gamma(n)$ is defined in terms of the integral of the function

$$f(x) = x^{n-1}e^{-x}, \quad n > 0.$$

Show that for any fixed value of n, the limit of $f(x)$ as x approaches infinity is zero.

47. The velocity of an object falling through a resisting medium such as air or water is given by

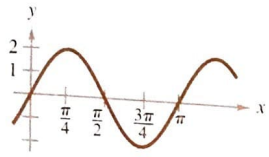
$$v = \frac{32}{k}\left(1 - e^{-kt} + \frac{v_0 k e^{-kt}}{32}\right)$$

where v_0 is the initial velocity, t is the time, and k is the resistance constant of the medium. Use L'Hôpital's Rule to find the formula for the velocity of a falling

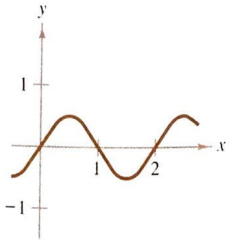
EXERCISES for Section 8.2

In Exercises 1–10, determine the period and amplitude of the given function.

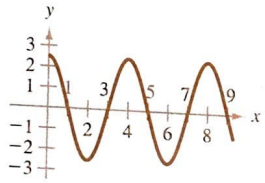
1. $y = 2 \sin 2x$



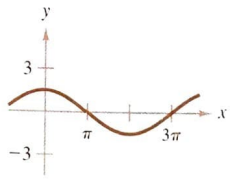
2. $y = \frac{1}{2} \sin \pi x$



3. $y = \frac{5}{2} \cos \frac{\pi x}{2}$



4. $y = \frac{3}{2} \cos \frac{x}{2}$



5. $y = -2 \sin \frac{x}{3}$

6. $y = -\cos \frac{2x}{3}$

7. $y = -2 \sin 10x$

8. $y = \frac{1}{2} \cos \frac{2x}{3}$

9. $y = 3 \sin 4\pi x$

10. $y = \frac{2}{3} \cos \frac{\pi x}{10}$

In Exercises 11 and 12, use a graphing utility to graph each function f on the same set of coordinate axes when $c = -2$, $c = -1$, $c = 1$, and $c = 2$. Write a description of the change in the graph caused by changing c.

11. (a) $f(x) = c \sin x$
(b) $f(x) = \cos(cx)$
(c) $f(x) = \cos(\pi x - c)$

12. (a) $f(x) = \sin x + c$
(b) $f(x) = -\sin(2\pi x - c)$
(c) $f(x) = c \cos x$

In Exercises 13–26, sketch the graph of the given function.

13. $y = \sin \frac{x}{2}$

14. $y = 2 \cos 2x$

15. $y = -2 \sin 6x$

16. $y = \cos 2\pi x$

17. $y = -\sin \frac{2\pi x}{3}$

18. $y = 2 \tan x$

19. $y = \csc \frac{x}{2}$

20. $y = \tan 2x$

21. $y = 2 \sec 2x$

22. $y = \csc 2\pi x$

23. $y = \sin (x + \pi)$

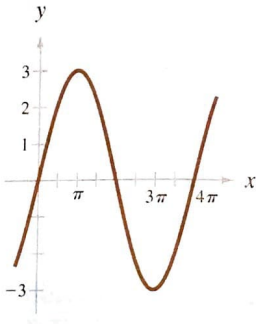
24. $y = \cos \left(2x - \frac{\pi}{3}\right)$

25. $y = 1 + \cos \left(x - \frac{\pi}{2}\right)$

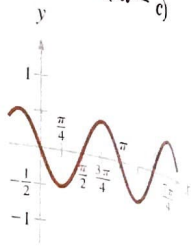
26. $y = -1 + \sin \left(x + \frac{\pi}{2}\right)$

In Exercises 27 and 28, find a, b, and c so that the graph of the function matches the graph in the figure.

27. $y = a \cos(bx - c)$



28. $y = a \sin(bx - c)$



In Exercises 29 and 30, use a graphing utility to compare the graph of f with the given graph. In each case, the graph of f is a Fourier approximation of the given graph. Try to improve the approximation by adding a term to $f(x)$. Use a graphing utility to verify that your new approximation is better than the original. Can you find other terms to add to make the approximation even better? What is the pattern? (In Exercise 29, sine terms can be used to improve the approximation and in Exercise 30, cosine terms can be used.)

29. $f(x) = \frac{4}{\pi}\left(\sin \pi x + \frac{1}{3} \sin 3\pi x\right)$

FIGURE FOR 29

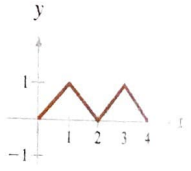


FIGURE FOR 30

30. $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi x + \frac{1}{9} \cos 3\pi x\right)$

In Exercises 31–48, determine the limit (if it exists).

31. $\lim_{x \to 0} \frac{\sin x}{5x}$

32. $\lim_{x \to 0} \frac{3(1 - \cos x)}{x}$

33. $\lim_{\theta \to 0} \frac{\sec \theta - 1}{\theta \sec \theta}$

34. $\lim_{\theta \to 0} \frac{\cos \theta \tan \theta}{\theta}$

35. $\lim_{x \to 0} \frac{\sin^2 x}{x}$

36. $\lim_{\phi \to \pi} \phi \sec \phi$

37. $\lim_{x \to \pi/2} \frac{\cos x}{\cot x}$

38. $\lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x}$

39. $\lim_{t \to 0} \frac{\sin^2 t}{t^2}$ [Hint: Find $\lim_{t \to 0} \left(\frac{\sin t}{t}\right)^2$.]

40. $\lim_{t \to 0} \frac{\sin 3t}{t}$ [Hint: Find $\lim_{t \to 0} 3\left(\frac{\sin 3t}{3t}\right)$.]

41. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$

[Hint: Find $\lim_{x\to 0} \left(\dfrac{2 \sin 2x}{2x}\right)\left(\dfrac{3x}{3 \sin 3x}\right)$.]

42. $\lim_{x\to 0} \dfrac{\tan^2 x}{x}$

43. $\lim_{h\to 0} \dfrac{(1 - \cos h)^2}{h}$

44. $\lim_{h\to 0} (1 + \cos 2h)$

45. $\lim_{x\to \pi} \cot x$

46. $\lim_{x\to \pi/2} \sec x$

47. $\lim_{x\to 0^+} \dfrac{2}{\sin x}$

48. $\lim_{x\to \pi/2^+} \dfrac{-2}{\cos x}$

In Exercises 49–54, find the discontinuities (if any) for the function. Which are removable?

49. $f(x) = x + \sin x$

50. $f(x) = \cos \dfrac{\pi x}{2}$

51. $f(x) = \csc 2x$

52. $f(x) = \tan \dfrac{\pi x}{2}$

53. $f(x) = \begin{cases} \csc \dfrac{\pi x}{6}, & |x - 3| < 2 \\ 2, & |x - 3| \geq 2 \end{cases}$

54. $f(x) = \begin{cases} \tan \dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \geq 1 \end{cases}$

Exercises 55 and 56, use a computer or graphics lculator to sketch the graph of the function f, and find e specified limit (if it exists).

$f(x) = \dfrac{\sin 5x}{\sin 2x}$, $\lim_{x\to 0} f(x)$

$f(x) = \dfrac{1 - \cos 3x}{2x}$, $\lim_{x\to 0} f(x)$

xercises 57–62, use a graphing utility to graph the n function and the equations $y = x$ and $y = -x$ in the same viewing rectangle. Using the graphs to observe the Squeeze Theorem visually, find $\lim_{x\to 0} f(x)$.

57. $f(x) = x \cos x$

58. $f(x) = |x \sin x|$

59. $f(x) = |x| \sin x$

60. $f(x) = |x| \cos x$

61. $f(x) = x \sin \dfrac{1}{x}$

62. $h(x) = x \cos \dfrac{1}{x}$

63. Use a graphing utility to graph $f(x) = x$, $g(x) = \sin x$, and $h(x) = \sin x/x$ in the same viewing rectangle. Compare the magnitudes of $f(x)$ and $g(x)$ when x is "close to" 0. Use the comparison to write a short paragraph explaining why $\lim_{x\to 0} h(x) = 1$.

64. Use a graphing utility to graph $f(x) = x$, $g(x) = \sin^2 x$, and $h(x) = \sin^2 x/x$ in the same viewing rectangle. Compare the magnitudes of $f(x)$ and $g(x)$ when x is "close to" 0. Use the comparison to write a short paragraph explaining why $\lim_{x\to 0} h(x) = 0$.

65. Find the limit as x approaches zero of the ratio of the triangle's area to the total shaded area in the figure.

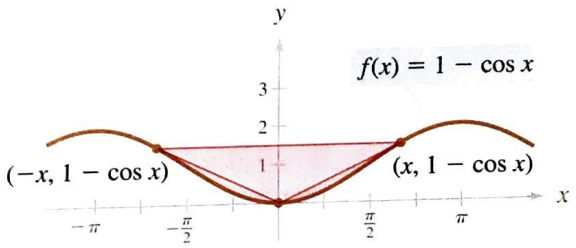


FIGURE FOR 65

66. When tuning a piano, a technician strikes a tuning fork for the A above middle C and sets up a wave motion that can be approximated by

$$y = 0.001 \sin 880\pi t$$

where t is the time in seconds.
(a) What is the period p of this function?
(b) What is the frequency f of this note? ($f = 1/p$)
(c) Sketch the graph of this function.

67. Prove the second part of Theorem 8.2.

Derivatives of Trigonometric Functions

atives of sine and cosine functions ▪ Derivatives of other trigonometric functions ▪ Applications

In the previous section, we discussed the following limits.

$$\lim_{\Delta x\to 0} \frac{\sin \Delta x}{\Delta x} = 1 \quad \text{and} \quad \lim_{\Delta x\to 0} \frac{1 - \cos \Delta x}{\Delta x} = 0$$

Choosing the positive solution, we have $x = 8$. Finally, when $\theta = \pi/3$ and $x = 8$, the velocity of the piston is

$$\frac{dx}{dt} = \frac{6(8)(\sqrt{3}/2)}{6(1/2) - 16}(400\pi) = \frac{9600\pi\sqrt{3}}{-13} \approx -4018 \text{ in./min.}$$

REMARK Note that the velocity in Example 12 is negative because the piston is moving to the left.

EXERCISES for Section 8.3

In Exercises 1–44, find the derivative of the given function.

1. $y = x^2 - \frac{1}{2}\cos x$
2. $y = 5 + \sin x$
3. $y = \frac{1}{x} - 3\sin x$
4. $g(t) = \pi \cos t$
5. $f(x) = 4\sqrt{x} + 3\cos x$
6. $f(x) = 2\sin x + 3\cos x$
7. $f(t) = t^2 \sin t$
8. $f(x) = \frac{\sin x}{x}$
9. $g(t) = \frac{\cos t}{t}$
10. $f(\theta) = (\theta + 1)\cos\theta$
11. $y = \tan x - x$
12. $y = x + \cot x$
13. $y = 5x \csc x$
14. $y = \frac{\sec x}{x}$
15. $f(\theta) = -\csc\theta - \sin\theta$
16. $h(x) = x\sin x + \cos x$
17. $g(t) = t^2 \sin t + 2t\cos t - 2\sin t$
18. $h(\theta) = 5\sec 3\theta + \tan 3\theta$
19. $f(x) = \sin \pi x \cos \pi x$
20. $f(x) = \tan 2x \cot 2x$
21. $y = \frac{1 + \csc x}{1 - \csc x}$
22. $y = \frac{\sin\theta}{1 - \cos\theta}$
23. $y = \cos 3x$
24. $y = \sin 2x$
25. $y = 3\tan 4x$
26. $y = 2\cos\frac{x}{2}$
27. $y = \sin \pi x$
28. $y = \sec x^2$
29. $y = \frac{1}{4}\sin^2 x$
30. $y = 5\cos^2 \pi x$
31. $y = \frac{1}{4}\sin^2 2x$
32. $y = 5\cos(\pi x)^2$
33. $y = \sqrt{\sin x}$
34. $y = \csc^2 4x$
35. $y = \sec^3 2x$
36. $y = x^2 \sin\frac{1}{x}$
37. $y = \ln|\csc x - \cot x|$
38. $y = \ln|\sec x + \tan x|$
39. $y = e^x(\sin x + \cos x)$
40. $y = \tan^2 e^x$
41. $y = e^{\tan x}$
42. $y = \ln|\cot x|$
43. $y = \ln|\tan x|$
44. $y = \ln|\sin x|$

In Exercises 45–50, use implicit differentiation to find dy/dx and evaluate the derivative at the indicated point.

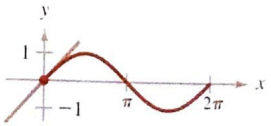
Equation	*Point*
45. $\sin x + \cos 2y = 2$	$\left(\frac{\pi}{2}, 0\right)$
46. $2\sin x \cos y = 1$	$\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$
47. $\tan(x + y) = x$	$(0, 0)$
48. $\cot y = x - y$	$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
49. $x\cos y = 1$	$\left(2, \frac{\pi}{3}\right)$
50. $x = \sec\frac{1}{y}$	$\left(\sqrt{2}, \frac{\pi}{4}\right)$

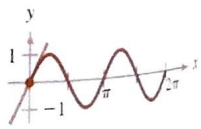
In Exercises 51 and 52, show that the function satisfies the differential equation.

51. $y = 2\sin x + 3\cos x$
$y'' + y = 0$
52. $y = e^x(\cos\sqrt{2}x + \sin\sqrt{2}x)$
$y'' - 2y' + 3y = 0$

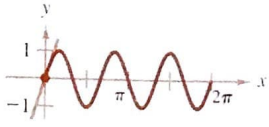
In Exercises 53 and 54, find the slope of the tangent line to the given sine function at the origin. Compare this value to the number of complete cycles in the interval $[0, 2\pi]$.

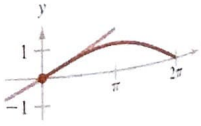
53. (a) $y = \sin x$ (b) $y = \sin 2x$

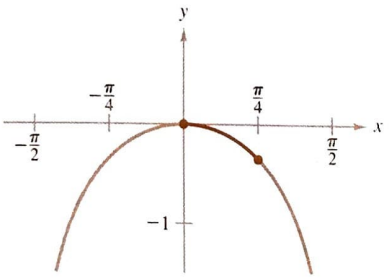




54. (a) $y = \sin 3x$ (b) $y = \sin\frac{x}{2}$







$y = \ln(\cos x)$

FIGURE 8.31

EXAMPLE 14 Finding arc length

Find the arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$, as shown in Figure 8.31.

SOLUTION

Since

$$y' = -\frac{\sin x}{\cos x} = -\tan x$$

the arc length is given by

$$s = \int_a^b \sqrt{1 + (y')^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \tan^2 x}\, dx = \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx$$
$$= \int_0^{\pi/4} \sec x\, dx = \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$
$$= \ln(\sqrt{2} + 1) - \ln(1) \approx 0.8814.$$

EXERCISES for Section 8.4

In Exercises 1–36, evaluate the given integral.

1. $\int (2\sin x + 3\cos x)\, dx$
2. $\int (t^2 - \sin t)\, dt$
3. $\int (1 - \csc t \cot t)\, dt$
4. $\int (\theta^2 + \sec^2\theta)\, d\theta$
5. $\int (\sec^2\theta - \sin\theta)\, d\theta$
6. $\int \sec y(\tan y - \sec y)\, dy$
7. $\int \sin 2x\, dx$
8. $\int \cos 6x\, dx$
9. $\int x\cos x^2\, dx$
10. $\int x\sin x^2\, dx$
11. $\int \sec^2\frac{x}{2}\, dx$
12. $\int \csc^2\frac{x}{2}\, dx$
13. $\int \frac{\csc^2 x}{\cot^3 x}\, dx$
14. $\int \frac{\sin x}{\cos^2 x}\, dx$
15. $\int \cot^2 x\, dx$
16. $\int \csc 2x \cot 2x\, dx$
17. $\int \tan^4 x \sec^2 x\, dx$
18. $\int \sqrt{\cot x}\csc^2 x\, dx$
19. $\int \cot \pi x\, dx$
20. $\int \tan 5x\, dx$
21. $\int \csc 2x\, dx$
22. $\int \sec\frac{x}{2}\, dx$
23. $\int \frac{\sec^2 x}{\tan x}\, dx$
24. $\int \frac{\tan^2 2x}{\sec 2x}\, dx$
25. $\int \frac{\sec x \tan x}{\sec x - 1}\, dx$
26. $\int \frac{\sin x}{1 + \cos x}\, dx$
27. $\int \frac{\cos t}{1 + \sin t}\, dt$
28. $\int \frac{\sin^2 x - \cos^2 x}{\cos x}\, dx$
29. $\int \frac{1 - \cos\theta}{\theta - \sin\theta}\, d\theta$
30. $\int \frac{1 - \sin^2\theta}{\cos^2\theta}\, d\theta$
31. $\int e^x \cos e^x\, dx$
32. $\int e^{\sin x}\cos x\, dx$
33. $\int e^{-x}\tan(e^{-x})\, dx$
34. $\int e^{\sec x}\sec x \tan x\, dx$
35. $\int (\sin 2x + \cos 2x)^2\, dx$
36. $\int (\csc 2\theta - \cot 2\theta)^2\, d\theta$

In Exercises 37–44, evaluate the definite integral.

37. $\int_0^{\pi/2} \cos\frac{2x}{3}\, dx$
38. $\int_0^{\pi/2} \sin 2x\, dx$
39. $\int_{\pi/2}^{2\pi/3} \sec^2\frac{x}{2}\, dx$
40. $\int_{\pi/2}^{\pi/2} (x + \cos x)\, dx$
41. $\int_{\pi/12}^{\pi/4} \csc 2x \cot 2x\, dx$
42. $\int_0^{\pi/8} \sin 2x \cos 2x\, dx$
43. $\int_0^1 \sec(1 - x)\tan(1 - x)\, dx$
44. $\int_0^{\pi/4} (\sec x)^3(\sec x \tan x)\, dx$

EXERCISES for Section 8.5

In Exercises 1–10, evaluate the given expression.

1. $\arcsin \frac{1}{2}$
2. $\arcsin 0$
3. $\arccos \frac{1}{2}$
4. $\arccos 0$
5. $\arctan \frac{\sqrt{3}}{3}$
6. $\operatorname{arccot}(-1)$
7. $\operatorname{arccsc} \sqrt{2}$
8. $\arcsin(-0.39)$
9. $\operatorname{arcsec} 1.269$
10. $\arctan(-3)$

In Exercises 11–16, evaluate the given expression without a calculator. [Hint: Make a sketch of a right triangle, as illustrated in Example 4.]

11. (a) $\sin\left(\arcsin \frac{1}{2}\right)$ (b) $\cos\left(2 \arcsin \frac{1}{2}\right)$
12. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$ (b) $\cos\left(\arcsin \frac{5}{13}\right)$
13. (a) $\sin\left(\arctan \frac{3}{4}\right)$ (b) $\sec\left(\arcsin \frac{4}{5}\right)$
14. (a) $\tan(\operatorname{arccot} 2)$ (b) $\cos(\operatorname{arcsec} \sqrt{5})$
15. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$ (b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$
16. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$ (b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right]$

In Exercises 17–26, write the given expression in algebraic form.

17. $\tan(\arctan x)$
18. $\sin(\arccos x)$
19. $\cos(\arcsin 2x)$
20. $\sec(\arctan 3x)$
21. $\sin(\operatorname{arcsec} x)$
22. $\cos(\operatorname{arccot} x)$
23. $\tan\left(\operatorname{arcsec} \frac{x}{3}\right)$
24. $\sec[\arcsin(x-1)]$
25. $\csc\left(\arctan \frac{x}{\sqrt{2}}\right)$
26. $\cos\left(\arcsin \frac{x-h}{r}\right)$

In Exercises 27 and 28, fill in the blank.

27. $\arctan \frac{9}{x} = \arcsin(\quad)$
28. $\arcsin \frac{\sqrt{36-x^2}}{6} = \arccos(\quad)$

In Exercises 29 and 30, verify each identity.

29. (a) $\operatorname{arccsc} x = \arcsin \frac{1}{x}, |x| \geq 1$
(b) $\operatorname{arccot} x = \arctan \frac{1}{x}, x > 0$
30. (a) $\arcsin(-x) = -\arcsin x, |x| \leq 1$
(b) $\arccos(-x) = \pi - \arccos x, |x| \leq 1$

In Exercises 31–34, sketch the graph of the function.

31. $f(x) = \arcsin(x-1)$
32. $f(x) = \arctan x + \frac{\pi}{2}$
33. $f(x) = \operatorname{arcsec} 2x$
34. $f(x) = \arccos \frac{x}{4}$

In Exercises 35–38, solve the given equation for x.

35. $\arcsin(3x - \pi) = \frac{1}{2}$
36. $\arctan 2x = -1$
37. $\arcsin \sqrt{2x} = \arccos \sqrt{x}$
38. $\arccos x = \operatorname{arcsec} x$

In Exercises 39–58, find the derivative of the given function.

39. $f(x) = \arcsin 2x$
40. $f(x) = \arcsin x^2$
41. $f(x) = 2 \arcsin(x-1)$
42. $f(x) = \arccos \sqrt{x}$
43. $f(x) = 3 \arccos \frac{x}{2}$
44. $f(x) = \arctan \sqrt{x}$
45. $f(x) = \arctan 5x$
46. $f(x) = x \arctan x$
47. $f(x) = \arccos \frac{1}{x}$
48. $f(x) = \operatorname{arcsec} 2x$
49. $f(x) = \arcsin x + \arccos x$
50. $f(x) = \operatorname{arcsec} x + \operatorname{arccsc} x$
51. $h(t) = \sin(\arccos t)$
52. $g(t) = \tan(\arcsin t)$
53. $f(t) = \frac{1}{\sqrt{6}} \arctan \frac{\sqrt{6}t}{2}$
54. $f(x) = \frac{1}{2}\left(\frac{1}{2} \ln \frac{x+1}{x-1} - \arctan x\right)$
55. $f(x) = \frac{1}{2}\left(\frac{1}{2} \ln \frac{x+1}{x-1} + \arctan x\right)$
56. $f(x) = \frac{1}{2}(x\sqrt{1-x^2} + \arcsin x)$
57. $f(x) = x \arcsin x + \sqrt{1-x^2}$
58. $f(x) = x \arctan 2x - \frac{1}{4} \ln(1+4x^2)$

In Exercises 59 and 60, find the point of inflection of the graph of the given function.

59. $f(x) = \arcsin x$
60. $f(x) = \operatorname{arccot} 2x$

In Exercises 61 and 62, find any relative extrema of the given function.

61. $f(x) = \operatorname{arcsec} x - x$
62. $f(x) = \arcsin x - 2x$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \text{arcsec}\,|x| + C$$

(b) We *can* also evaluate this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2-1}} &= \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx \\ &= \frac{1}{2}\left(\frac{(x^2-1)^{1/2}}{1/2}\right) + C \\ &= \sqrt{x^2-1} + C\end{aligned}$$

(c) We *cannot* evaluate this integral using our present techniques. (You should scan the list of Basic Integration Formulas to verify this conclusion.)

□

EXAMPLE 9 Comparing integration problems

Evaluate as many of the following integrals as you can using the formulas and techniques we have studied so far in the text.

(a) $\displaystyle\int \frac{dx}{x\ln x}$ (b) $\displaystyle\int \frac{\ln x\,dx}{x}$ (c) $\displaystyle\int \ln x\,dx$

SOLUTION

(a) We *can* evaluate this integral (it fits the log formula).

$$\int \frac{dx}{x\ln x} = \int \frac{1/x}{\ln x}\,dx = \ln|\ln x| + C$$

(b) We *can* also evaluate this integral (it fits the Power Rule).

$$\int \frac{\ln x}{x}\,dx = \int \left(\frac{1}{x}\right)(\ln x)^1\,dx = \frac{(\ln x)^2}{2} + C$$

(c) We *cannot* evaluate this integral using our present techniques. □

REMARK Note in Examples 8 and 9 that it is the *simplest* function that we cannot yet integrate.

EXERCISES for Section 8.6

In Exercises 1–26, evaluate the given integral.

e **1.** $\displaystyle\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx$ **2.** $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ **3.** $\displaystyle\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx$ **4.** $\displaystyle\int_{\sqrt{3}}^3 \frac{1}{9+x^2}\,dx$

5. $\displaystyle\int \frac{1}{x\sqrt{4x^2-1}}\,dx$ **6.** $\displaystyle\int \frac{1}{4+(x-1)^2}\,dx$

7. $\int \frac{x^3}{x^2+1}\,dx$

8. $\int \frac{x^4-1}{x^2+1}\,dx$

9. $\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$

10. $\int \frac{t}{t^4+16}\,dt$

11. $\int \frac{t}{\sqrt{1-t^4}}\,dt$

12. $\int \frac{1}{x\sqrt{x^4-4}}\,dx$

13. $\int \frac{\arctan x}{1+x^2}\,dx$

14. $\int \frac{1}{(x-1)\sqrt{(x-1)^2-4}}\,dx$

15. $\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

16. $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

17. $\int_{-1/2}^{0} \frac{x}{\sqrt{1-x^2}}\,dx$

18. $\int_{-\sqrt{3}}^{0} \frac{x}{1+x^2}\,dx$

19. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

20. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$

21. $\int \frac{1}{9+(x-3)^2}\,dx$

22. $\int \frac{x+1}{x^2+1}\,dx$

23. $\int \frac{1}{\sqrt{x}(1+x)}\,dx$

24. $\int_1^2 \frac{1}{3+(x-2)^2}\,dx$

25. $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$

26. $\int \frac{e^{2x}}{4+e^{4x}}\,dx$

In Exercises 27–40, evaluate the given integral. (Complete the square, if necessary.)

27. $\int_0^2 \frac{dx}{x^2-2x+2}$

28. $\int_{-3}^{-1} \frac{dx}{x^2+6x+13}$

29. $\int \frac{2x}{x^2+6x+13}\,dx$

30. $\int \frac{2x-5}{x^2+2x+2}\,dx$

31. $\int \frac{1}{\sqrt{-x^2-4x}}\,dx$

32. $\int \frac{1}{\sqrt{-x^2+2x}}\,dx$

33. $\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx$

34. $\int \frac{x-1}{\sqrt{x^2-2x}}\,dx$

35. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx$

36. $\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx$

37. $\int \frac{x}{x^4+2x^2+2}\,dx$

38. $\int \frac{x}{\sqrt{9+8x^2-x^4}}\,dx$

39. $\int \frac{1}{\sqrt{-16x^2+16x-3}}\,dx$

40. $\int \frac{1}{(x-1)\sqrt{9x^2-18x+5}}\,dx$

In Exercises 41–44, use substitution to evaluate the given integral.

41. $\int \frac{\sqrt{x-1}}{x}\,dx$

42. $\int \frac{\sqrt{x-2}}{x+1}\,dx$

43. $\int \sqrt{e^t-3}\,dt$

44. $\int \frac{1}{t\sqrt{t}+\sqrt{t}}\,dt$

In Exercises 45–48, find the area of the region bounded by the graphs of the given equations.

45. $y = \frac{1}{1+x^2},\ y = 0,\ x = 0,\ x = 1$

46. $y = \frac{1}{\sqrt{4-x^2}},\ y = 0,\ x = 0,\ x = 1$

47. $y = \frac{1}{x^2-2x+5},\ y = 0,\ x = 1,\ x = 3$

48. $y = \frac{1}{\sqrt{3+2x-x^2}},\ y = 0,\ x = 0,\ x = 2$

49. An object is projected upward with an initial velocity of 500 feet per second. If the air resistance is proportional to the square of the velocity, we obtain the equation

$$\frac{dv}{dt} = -(32 + kv^2)$$

where 32 feet per second is the acceleration due to gravity and k is a constant. Find the velocity (on the upward flight) as a function of time by solving the equation

$$\int \frac{dv}{32+kv^2} = -\int dt.$$

50. A weight of mass m is attached to a spring and oscillates with simple harmonic motion (see figure). By Hooke's Law, we can determine that

$$\int \frac{dy}{\sqrt{A^2-y^2}} = \int \sqrt{\frac{k}{m}}\,dt$$

where A is the maximum displacement, t is the time, and k is a constant. Find y as a function of t, given that $y = 0$ when $t = 0$.

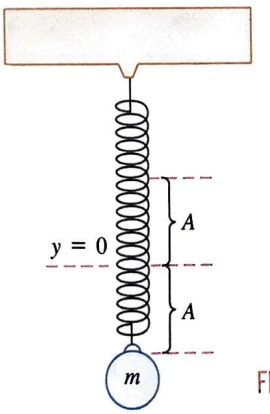


FIGURE FOR 50

51. (a) Show that

$$\int_0^1 \frac{4}{1+x^2}\,dx = \pi.$$

(b) Approximate the number π using Simpson's Rule (with $n = 6$) and the integral of part (a).

PROCEDURES FOR FITTING INTEGRANDS TO BASIC FORMULAS

Technique	*Example*
Expand (numerator)	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$
Separate numerator	$\dfrac{1 + x}{x^2 + 1} = \dfrac{1}{x^2 + 1} + \dfrac{x}{x^2 + 1}$
Complete the square	$\dfrac{1}{\sqrt{2x - x^2}} = \dfrac{1}{\sqrt{1 - (x - 1)^2}}$
Divide if rational function is improper	$\dfrac{x^2}{x^2 + 1} = 1 - \dfrac{1}{x^2 + 1}$
Add and subtract terms in numerator	$\dfrac{2x}{x^2 + 2x + 1} = \dfrac{2x + 2 - 2}{x^2 + 2x + 1} = \dfrac{2x + 2}{x^2 + 2x + 1} - \dfrac{2}{(x + 1)^2}$
Use trigonometric identities	$\cot^2 x = \csc^2 x - 1$
Multiply and divide by Pythagorean conjugate	$\dfrac{1}{1 + \sin x} = \left(\dfrac{1}{1 + \sin x}\right)\left(\dfrac{1 - \sin x}{1 - \sin x}\right) = \dfrac{1 - \sin x}{1 - \sin^2 x} = \dfrac{1 - \sin x}{\cos^2 x} = \sec^2 x - \dfrac{\sin x}{\cos^2 x}$

REMARK Watch out for this common error when fitting integrands to basic fomulas.

$$\frac{1}{x^2 + 1} \neq \frac{1}{x^2} + \frac{1}{1}$$ Do *not* separate denominators

EXERCISES for Section 9.1

In Exercises 1–50, evaluate the indefinite integral.

1. $\int (3x - 2)^4\, dx$
2. $\int \dfrac{2}{(t - 9)^2}\, dt$
3. $\int (-2x + 5)^{3/2}\, dx$
4. $\int x\sqrt{4 - 2x^2}\, dx$
5. $\int \left[v + \dfrac{1}{(3v - 1)^3}\right] dv$
6. $\int \dfrac{2t - 1}{t^2 - t + 2}\, dt$
7. $\int \dfrac{t^2 - 3}{-t^3 + 9t + 1}\, dt$
8. $\int \dfrac{2x}{x - 4}\, dx$
9. $\int \dfrac{x^2}{x - 1}\, dx$
10. $\int \dfrac{x + 1}{\sqrt{x^2 + 2x - 4}}\, dx$
11. $\int \left(\dfrac{1}{3x - 1} - \dfrac{1}{3x + 1}\right) dx$
12. $\int \dfrac{e^x}{1 + e^x}\, dx$
13. $\int t \sin t^2\, dt$
14. $\int \sec 4u\, du$
15. $\int \cos x\, e^{\sin x}\, dx$
16. $\int e^{5x}\, dx$
17. $\int \dfrac{(1 + e^t)^2}{e^t}\, dt$
18. $\int \dfrac{1 + \sin x}{\cos x}\, dx$
19. $\int \sec 3x \tan 3x\, dx$
20. $\int \dfrac{1}{\sqrt{x}(1 - 2\sqrt{x})}\, dx$
21. $\int \dfrac{2}{e^{-x} + 1}\, dx$
22. $\int \dfrac{1}{2e^x - 3}\, dx$
23. $\int \dfrac{1}{1 - \cos x}\, dx$
24. $\int \dfrac{1}{\sec x - 1}\, dx$
25. $\int \dfrac{2t - 1}{t^2 + 4}\, dt$
26. $\int \dfrac{2}{(2t - 1)^2 + 4}\, dt$
27. $\int \dfrac{3}{t^2 + 1}\, dt$
28. $\int \dfrac{3}{\sqrt{1 - t^2}}\, dt$
29. $\int \dfrac{1}{x\sqrt{x^2 - 4}}\, dx$
30. $\int \dfrac{-2x}{\sqrt{x^2 - 4}}\, dx$
31. $\int \dfrac{-1}{\sqrt{1 - (2t - 1)^2}}\, dt$
32. $\int \dfrac{1}{4 + 3x^2}\, dx$

Finally, the solution is given by multiplying the *signed* products of the diagonal entries to obtain

$$\int x^2 \sin 4x\, dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x + C.$$

EXERCISES for Section 9.2

In Exercises 1–34, evalute the given integral. (Note: Solve by the simplest method—not all require integration by parts.)

1. $\int xe^{2x}\, dx$
2. $\int x^2e^{2x}\, dx$
3. $\int xe^{x^2}\, dx$
4. $\int x^2e^{x^3}\, dx$
5. $\int xe^{-2x}\, dx$
6. $\int \frac{x}{e^x}\, dx$
7. $\int x^3e^x\, dx$
8. $\int \frac{e^{1/t}}{t^2}\, dt$
9. $\int x^3 \ln x\, dx$
10. $\int \ln x\, dx$
11. $\int t \ln (t+1)\, dt$
12. $\int \frac{1}{x(\ln x)^3}\, dx$
13. $\int (\ln x)^2\, dx$
14. $\int \ln 3x\, dx$
15. $\int \frac{(\ln x)^2}{x}\, dx$
16. $\int \frac{\ln x}{x^2}\, dx$
17. $\int \frac{xe^{2x}}{(2x+1)^2}\, dx$
18. $\int \frac{x^3e^{x^2}}{(x^2+1)^2}\, dx$
19. $\int (x^2-1)e^x\, dx$
20. $\int \frac{\ln 2x}{x^2}\, dx$
21. $\int x\sqrt{x-1}\, dx$
22. $\int x^2\sqrt{x-1}\, dx$
23. $\int \frac{x^2}{\sqrt{2+3x}}\, dx$
24. $\int \frac{x}{\sqrt{2+3x}}\, dx$
25. $\int x \cos x\, dx$
26. $\int x^2 \cos x\, dx$
27. $\int x \sec^2 x\, dx$
28. $\int \theta \sec\theta \tan\theta\, d\theta$
29. $\int \arcsin 2x\, dx$
30. $\int \arccos x\, dx$
31. $\int \arctan x\, dx$
32. $\int \arctan \frac{x}{2}\, dx$
33. $\int e^{2x} \sin x\, dx$
34. $\int e^x \cos 2x\, dx$

In Exercises 35–40, evaluate the definite integral.

35. $\int_0^{\pi} x \sin 2x\, dx$
36. $\int_0^1 x \arcsin x^2\, dx$
37. $\int_0^1 e^x \sin x\, dx$
38. $\int_0^1 x^2e^x\, dx$
39. $\int_0^{\pi/2} x \cos x\, dx$
40. $\int_0^1 \ln (1+x^2)\, dx$

In Exercises 41–46, use the tabular method for repeated applications of integration by parts to evaluate the given integral.

41. $\int x^2e^{2x}\, dx$
42. $\int x^4e^{-x}\, dx$
43. $\int x^3 \sin x\, dx$
44. $\int x^3 \cos 2x\, dx$
45. $\int x \sec^2 x\, dx$
46. $\int x^2(x-2)^{3/2}\, dx$

47. Integrate $\int 2x\sqrt{2x-3}\, dx$
    (a) by parts, letting $dv = \sqrt{2x-3}\, dx$
    (b) by substitution, letting $u = \sqrt{2x-3}$.
48. Integrate $\int x\sqrt{4+x}\, dx$
    (a) by parts, letting $dv = \sqrt{4+x}\, dx$
    (b) by substitution, letting $u = \sqrt{4+x}$.

In Exercises 49 and 50, use a symbolic integration utility to evaluate the integral for $n = 1, 2, 3, 4$. Use the result to obtain a general rule for the integral for any positive integer n and test your result for $n = 5$.

49. $\int x^n \ln x\, dx$
50. $\int x^ne^x\, dx$

In Exercises 51–56, use integration by parts to verify the given formula. (Assume n is a positive integer.)

51. $\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$
52. $\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$
53. $\int x^n \ln x\, dx = \frac{x^{n+1}}{(n+1)^2}[-1 + (n+1)\ln x] + C$
54. $\int x^ne^{ax}\, dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx$
55. $\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2} + C$
56. $\int e^{ax} \cos bx\, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2} + C$

EXAMPLE 10 Using product-to-sum identities

Evaluate $\int \sin 5x \cos 4x\, dx$.

SOLUTION

Considering the second product-to-sum identity, we write

$$\begin{aligned}\int \sin 5x \cos 4x\, dx &= \frac{1}{2}\int (\sin 9x + \sin x)\, dx \\ &= \frac{1}{2}\left[-\frac{\cos 9x}{9} - \cos x\right] + C \\ &= -\frac{\cos 9x}{18} - \frac{\cos x}{2} + C.\end{aligned}$$

As you review this section, concentrate on the *general pattern* followed in evaluating trigonometric integrals.

EXERCISES for Section 9.3

In Exercises 1–50, evaluate the given integral.

1. $\int \cos^3 x \sin x\, dx$
2. $\int \cos^3 x \sin^2 x\, dx$
3. $\int \sin^5 2x \cos 2x\, dx$
4. $\int \sin^3 x\, dx$
5. $\int \sin^5 x \cos^2 x\, dx$
6. $\int \cos^3 \frac{x}{3}\, dx$
7. $\int \cos^2 3x\, dx$
8. $\int \sin^2 2x\, dx$
9. $\int \sin^4 \pi x\, dx$
10. $\int \cos^4 \frac{x}{2}\, dx$
11. $\int \sin^2 x \cos^2 x\, dx$
12. $\int \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\, dx$
13. $\int x \sin^2 x\, dx$ (Integration by parts)
14. $\int x^2 \sin^2 x\, dx$ (Integration by parts)
15. $\int \sec 3x\, dx$
16. $\int \sec^2 (2x - 1)\, dx$
17. $\int \sec^3 5x\, dx$
18. $\int \sec^6 \frac{x}{2}\, dx$
19. $\int \sec^3 \pi x\, dx$ (Integration by parts)
20. $\int \sec^5 \pi x\, dx$ (Integration by parts)
21. $\int \tan^3 (1 - x)\, dx$
22. $\int \tan^2 x\, dx$
23. $\int \tan^5 \frac{x}{4}\, dx$
24. $\int \tan^3 \frac{\pi x}{2} \sec^2 \frac{\pi x}{2}\, dx$
25. $\int \sec^2 x \tan x\, dx$
26. $\int \csc^2 3x \cot 3x\, dx$
27. $\int \tan^2 x \sec^2 x\, dx$
28. $\int \tan^5 2x \sec^2 2x\, dx$
29. $\int \sec^5 \pi x \tan \pi x\, dx$
30. $\int \sec^4 (1 - x) \tan (1 - x)\, dx$
31. $\int \sec^6 4x \tan 4x\, dx$
32. $\int \sec^2 \frac{x}{2} \tan \frac{x}{2}\, dx$
33. $\int \sec^3 x \tan x\, dx$
34. $\int \tan^3 3x\, dx$
35. $\int \tan^3 3x \sec 3x\, dx$
36. $\int \sqrt{\tan x} \sec^4 x\, dx$
37. $\int \cot^3 2x\, dx$
38. $\int \tan^4 \frac{x}{2} \sec^4 \frac{x}{2}\, dx$
39. $\int \csc^4 \theta\, d\theta$
40. $\int \tan^3 t \sec^3 t\, dt$
41. $\int \frac{\cot^2 t}{\csc t}\, dt$
42. $\int \frac{\cot^3 t}{\csc t}\, dt$
43. $\int \sin 3x \cos 2x\, dx$
44. $\int \cos 3\theta \cos(-2\theta)\, d\theta$
45. $\int \sin \theta \sin 3\theta\, d\theta$
46. $\int x \tan^2 x^2\, dx$
47. $\int \frac{1}{\sec x \tan x}\, dx$
48. $\int \frac{\sin^2 x - \cos^2 x}{\cos x}\, dx$
49. $\int (\tan^4 t - \sec^4 t)\, dt$
50. $\int \frac{1 - \sec t}{\cos t - 1}\, dt$

EXERCISES for Section 9.4

In Exercises 1–4, evaluate the indefinite integral using the substitution $x = 5 \sin \theta$.

1. $\int \frac{1}{(25 - x^2)^{3/2}}\,dx$
2. $\int \frac{1}{x^2\sqrt{25 - x^2}}\,dx$
3. $\int \frac{\sqrt{25 - x^2}}{x}\,dx$
4. $\int \frac{x^2}{\sqrt{25 - x^2}}\,dx$

In Exercises 5–8, evaluate the indefinite integral using the substitution $x = 2 \sec \theta$.

5. $\int \frac{1}{\sqrt{x^2 - 4}}\,dx$
6. $\int \frac{\sqrt{x^2 - 4}}{x}\,dx$
7. $\int x^3\sqrt{x^2 - 4}\,dx$
8. $\int \frac{x^3}{\sqrt{x^2 - 4}}\,dx$

In Exercises 9–12, evaluate the indefinite integral using the substitution $x = \tan \theta$.

9. $\int x\sqrt{1 + x^2}\,dx$
10. $\int \frac{x^3}{\sqrt{1 + x^2}}\,dx$
11. $\int \frac{1}{(1 + x^2)^2}\,dx$
12. $\int \frac{x^2}{(1 + x^2)^2}\,dx$

In Exercises 13–40, evaluate the indefinite integral.

13. $\int \frac{x}{\sqrt{x^2 + 9}}\,dx$
14. $\int \frac{1}{\sqrt{25 - x^2}}\,dx$
15. $\int_0^2 \sqrt{16 - 4x^2}\,dx$
16. $\int_0^2 x\sqrt{16 - 4x^2}\,dx$
17. $\int \frac{1}{\sqrt{x^2 - 9}}\,dx$
18. $\int \frac{t}{(1 - t^2)^{3/2}}\,dt$
19. $\int_0^{\sqrt{3}/2} \frac{t^2}{(1 - t^2)^{3/2}}\,dt$
20. $\int_0^{\sqrt{3}/2} \frac{1}{(1 - t^2)^{5/2}}\,dt$
21. $\int \frac{\sqrt{1 - x^2}}{x^4}\,dx$
22. $\int \frac{\sqrt{4x^2 + 9}}{x^4}\,dx$
23. $\int \frac{1}{x\sqrt{4x^2 + 9}}\,dx$
24. $\int \frac{1}{(x^2 + 3)^{3/2}}\,dx$
25. $\int \frac{x}{(x^2 + 3)^{3/2}}\,dx$
26. $\int \frac{1}{x\sqrt{4x^2 + 16}}\,dx$
27. $\int e^{2x}\sqrt{1 + e^{2x}}\,dx$
28. $\int \frac{1}{\sqrt{4x - x^2}}\,dx$
29. $\int (x + 1)\sqrt{x^2 + 2x + 2}\,dx$
30. $\int \frac{x^2}{\sqrt{2x - x^2}}\,dx$
31. $\int \frac{x}{\sqrt{x^2 + 4x + 8}}\,dx$
32. $\int \frac{x}{\sqrt{x^2 - 6x + 5}}\,dx$
33. $\int e^x\sqrt{1 - e^{2x}}\,dx$
34. $\int \frac{\sqrt{1 - x}}{\sqrt{x}}\,dx$
35. $\int \frac{1}{4 + 4x^2 + x^4}\,dx$
36. $\int \frac{x^3 + x + 1}{x^4 + 2x^2 + 1}\,dx$
37. $\int \sqrt{4 + 9x^2}\,dx$
38. $\int \sqrt{1 + x^2}\,dx$
39. $\int \operatorname{arcsec} 2x\,dx$
40. $\int x \arcsin x\,dx$

In Exercises 41–44, use a symbolic integration utility to evaluate the integral.

41. $\int \frac{x^2}{\sqrt{x^2 + 10x + 9}}\,dx$
42. $\int (x^2 + 2x + 11)^{3/2}\,dx$
43. $\int \frac{x^2}{\sqrt{x^2 - 1}}\,dx$
44. $\int x^2\sqrt{x^2 - 4}\,dx$

In Exercises 45 and 46, evaluate the given integral using (a) the given integration limits and (b) the limits obtained by trigonometric substitution.

45. $\int_0^3 \frac{x^3}{\sqrt{x^2 + 9}}\,dx$
46. $\int_0^{5/3} \sqrt{25 - 9x^2}\,dx$

47. Find the value of the definite integral of Example 4 by converting back to the variable x and evaluating the antiderivative at the original limits.

48. The field strength H of a magnet of length $2L$ on a particle r units from the center of the magnet is given by

$$H = \frac{2mL}{(r^2 + L^2)^{3/2}}$$

where $\pm m$ are the poles of the magnet (see figure). Find the average field strength as the particle moves from 0 to R units from the center by evaluating the integral

$$\frac{1}{R}\int_0^R \frac{2mL}{(r^2 + L^2)^{3/2}}\,dr.$$

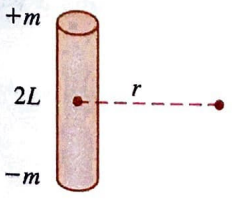


FIGURE FOR 48

In Exercises 49 and 50, find the fluid force on the circular observation window of radius 1 foot in a vertical wall of a large water-filled tank at a fish hatchery.

49. The center of the window is 3 feet below the surface of the water (see figure).

If $x = 2$, then in the basic equation we have $10 = 10A$ and $A = 1$. Now, from the expanded equation, we have the system

$$\begin{aligned} A + B &= 0 \\ 2A - 2B + C &= 3 \\ 2A - 2C &= 4. \end{aligned}$$

Using $A = 1$, we have $B = -1$ and $C = -1$. Therefore,

$$\begin{aligned} \int \frac{3x+4}{x^3-2x-4}\,dx &= \int \left(\frac{A}{x-2} + \frac{Bx+C}{x^2+2x+2}\right) dx \\ &= \int \left(\frac{1}{x-2} + \frac{-x-1}{x^2+2x+2}\right) dx \\ &= \ln|x-2| - \frac{1}{2}\ln(x^2+2x+2) + C. \end{aligned}$$

Before we conclude this section, here are a few things you should remember. First, it is not necessary to use the partial fractions technique on all rational functions. For instance, the following integral is evaluated more easily by the Log Rule.

$$\int \frac{x^2+1}{x^3+3x-4}\,dx = \frac{1}{3}\int \frac{3x^2+3}{x^3+3x-4}\,dx = \frac{1}{3}\ln|x^3+3x-4| + C$$

Second, if the integrand is not in reduced form, reducing it may eliminate the need for partial fractions, as shown in the following integral.

$$\begin{aligned} \int \frac{x^2-x-2}{x^3-2x-4}\,dx &= \int \frac{(x+1)(x-2)}{(x-2)(x^2+2x+2)}\,dx = \int \frac{x+1}{x^2+2x+2}\,dx \\ &= \frac{1}{2}\ln|x^2+2x+2| + C \end{aligned}$$

Finally, partial fractions can be used with some quotients involving transcendental functions. For instance, the substitution $u = \sin x$ allows us to write

$$\begin{aligned} \int \frac{\cos x}{\sin x(\sin x - 1)}\,dx &= \int \frac{du}{u(u-1)} = \int \left(-\frac{1}{u} + \frac{1}{u-1}\right) du \\ &= -\ln|u| + \ln|u-1| + C = \ln\left|\frac{\sin x - 1}{\sin x}\right| + C. \end{aligned}$$

EXERCISES for Section 9.5

In Exercises 1–30, evaluate the indefinite integral.

1. $\displaystyle\int \frac{1}{x^2-1}\,dx$
2. $\displaystyle\int \frac{1}{4x^2-9}\,dx$
3. $\displaystyle\int \frac{3}{x^2+x-2}\,dx$
4. $\displaystyle\int \frac{x+1}{x^2+4x+3}\,dx$
5. $\displaystyle\int \frac{5-x}{2x^2+x-1}\,dx$
6. $\displaystyle\int \frac{3x^2-7x-2}{x^3-x}\,dx$
7. $\displaystyle\int \frac{x^2+12x+12}{x^3-4x}\,dx$
8. $\displaystyle\int \frac{x^3-x+3}{x^2+x-2}\,dx$
9. $\displaystyle\int \frac{2x^3-4x^2-15x+5}{x^2-2x-8}\,dx$
10. $\displaystyle\int \frac{x+2}{x^2-4x}\,dx$
11. $\displaystyle\int \frac{4x^2+2x-1}{x^3+x^2}\,dx$
12. $\displaystyle\int \frac{2x-3}{(x-1)^2}\,dx$

13. $\int \frac{x^4}{(x-1)^3}\,dx$

14. $\int \frac{4x^2-1}{(2x)(x^2+2x+1)}\,dx$

15. $\int \frac{3x}{x^2-6x+9}\,dx$

16. $\int \frac{6x^2+1}{x^2(x-1)^3}\,dx$

17. $\int \frac{x^2-1}{x^3+x}\,dx$

18. $\int \frac{x}{x^3-1}\,dx$

19. $\int \frac{x^2}{x^4-2x^2-8}\,dx$

20. $\int \frac{2x^2+x+8}{(x^2+4)^2}\,dx$

21. $\int \frac{x}{16x^4-1}\,dx$

22. $\int \frac{x^2-4x+7}{x^3-x^2+x+3}\,dx$

23. $\int \frac{x^2+x+2}{(x^2+2)^2}\,dx$

24. $\int \frac{x^3}{(x^2-4)^2}\,dx$

25. $\int \frac{x^2+5}{x^3-x^2+x+3}\,dx$

26. $\int \frac{x^2+x+3}{x^4+6x^2+9}\,dx$

27. $\int \frac{6x^2-3x+14}{x^3-2x^2+4x-8}\,dx$

28. $\int \frac{x(2x-9)}{x^3-6x^2+12x-8}\,dx$

29. $\int \frac{2x^2-2x+3}{x^3-x^2-x-2}\,dx$

30. $\int \frac{x^3-6x^2+x-4}{1-x^4}\,dx$

In Exercises 31–36, evaluate the definite integral.

31. $\int_0^1 \frac{3}{2x^2+5x+2}\,dx$

32. $\int_3^4 \frac{1}{x^2-4}\,dx$

33. $\int_1^2 \frac{x+1}{x(x^2+1)}\,dx$

34. $\int_1^5 \frac{x-1}{x^2(x+1)}\,dx$

35. $\int_2^3 \frac{x^2-x+2}{x^3-x^2+x-1}\,dx$

36. $\int_0^1 \frac{x^2-x}{x^2+x+1}\,dx$

In Exercises 37–42, find the indefinite integral by using the indicated substitution.

37. $\int \frac{\sin x}{\cos x\,(\cos x-1)}\,dx$; let $u = \cos x$

38. $\int \frac{\sin x}{\cos x+\cos^2 x}\,dx$; let $u = \cos x$

39. $\int \frac{3\cos x}{\sin^2 x+\sin x-2}\,dx$; let $u = \sin x$

40. $\int \frac{\sec^2 x}{\tan x(\tan x+1)}\,dx$; let $u = \tan x$

41. $\int \frac{e^x}{(e^x-1)(e^x+4)}\,dx$; let $u = e^x$

42. $\int \frac{e^x}{(e^{2x}+1)(e^x-1)}\,dx$; let $u = e^x$

In Exercises 43–46, use the method of partial fractions to verify the given indefinite integral.

43. $\int \frac{1}{x(a+bx)}\,dx = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$

44. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

45. $\int \frac{x}{(a+bx)^2}\,dx = \frac{1}{b^2}\left(\frac{a}{a+bx} + \ln|a+bx|\right) + C$

46. $\int \frac{1}{x^2(a+bx)}\,dx = -\frac{1}{ax} - \frac{b}{a^2}\ln\left|\frac{x}{a+bx}\right| + C$

47. Find the area of the region bounded by the graphs of $y = 7/(16-x^2)$ and $y = 1$.

48. Find the centroid of the region bounded by the graphs of $y = 2x/(x^2+1)$, $y = 0$, $x = 0$, and $x = 3$.

49. Find the volume of the solid generated by revolving the region in Exercise 48 about the x-axis.

50. In Section 7.7 the exponential growth equation was derived from the assumption that the rate of growth is proportional to the existing quantity. In practice, there often exists some upper limit L past which growth cannot occur. In such cases, we assume the rate of growth to be proportional not only to the existing quantity, but also to the difference between the existing quantity y and the upper limit L. That is,

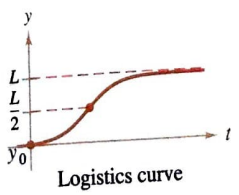
$$\frac{dy}{dt} = ky(L-y).$$

In integral form we can express this relationship as

$$\int \frac{dy}{y(L-y)} = \int k\,dt.$$

(a) Use partial fractions to evaluate the integral on the left, and solve for y as a function of t, where y_0 is the initial quantity.

(b) The graph of the function y is called a **logistics curve** (see figure). Show that the rate of growth is a maximum at the point of inflection, and that this occurs when $y = L/2$.



Logistics curve

FIGURE FOR 50

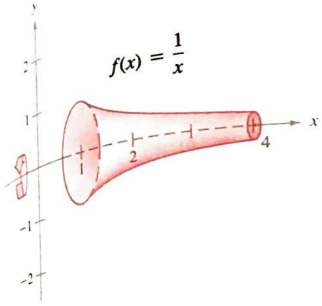


FIGURE 9.27

EXAMPLE 14 An application involving a solid of revolution

The solid formed by revolving the *unbounded* region lying between the graph of $f(x) = 1/x$ and the x-axis ($x \geq 1$) is called **Gabriel's Horn.** (See Figure 9.27.) Show that this solid has a finite volume and an infinite surface area.

SOLUTION

Using the disc method, with Theorem 9.3, we determine the volume to be

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx$$
$$= \pi\left(\frac{1}{2-1}\right) = \pi.$$

The surface area is given by

$$S = 2\pi \int_1^\infty f(x)\sqrt{1 + [f'(x)]^2}\, dx$$
$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\, dx$$
$$= 2\pi \int_1^\infty \frac{\sqrt{1 + x^4}}{x^3}\, dx.$$

REMARK For more information on solids that have a finite volume and an infinite surface area, see the article "Supersolids: Solids Having Finite Volume and Infinite Surfaces" by William P. Love in the January, 1989, issue of *Mathematics Teacher*.

Now, using the substitution $u = x^2$ together with Formula 30 in the integration table, we have

$$S = \lim_{b\to\infty} \pi\left[\frac{-\sqrt{1 + x^4}}{x^2} + \ln|x^2 + \sqrt{1 + x^4}|\right]_1^b.$$

We leave it up to you to show that this limit is infinite. ▭

EXERCISES for Section 9.7

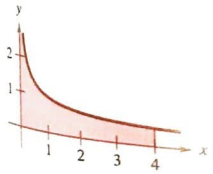
In Exercises 1–32, determine the divergence or convergence of the given improper integral. Evaluate the integral if it converges.

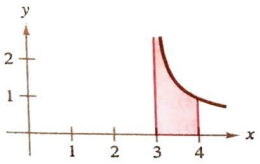
1. $\int_0^4 \frac{1}{\sqrt{x}}\, dx$

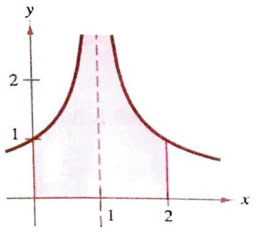
2. $\int_3^4 \frac{1}{\sqrt{x-3}}\, dx$

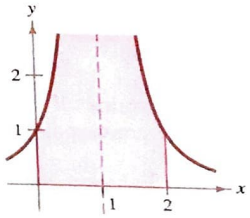
3. $\int_0^2 \frac{1}{(x-1)^{2/3}}\, dx$

4. $\int_0^2 \frac{1}{(x-1)^2}\, dx$



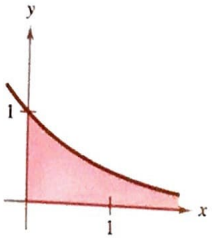


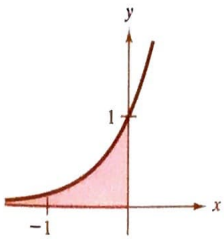




5. $\int_0^\infty e^{-x}\,dx$

6. $\int_{-\infty}^0 e^{2x}\,dx$





7. $\int_{-\infty}^0 xe^{-2x}\,dx$

8. $\int_0^\infty xe^{-x}\,dx$

9. $\int_0^\infty x^2e^{-x}\,dx$

10. $\int_0^\infty (x-1)e^{-x}\,dx$

11. $\int_1^\infty \frac{1}{x^2}\,dx$

12. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

13. $\int_0^\infty e^{-x}\cos x\,dx$

14. $\int_0^\infty e^{-ax}\sin bx\,dx,\ a>0$

15. $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

16. $\int_0^\infty \frac{x^3}{(x^2+1)^2}\,dx$

17. $\int_0^\infty \frac{1}{e^x+e^{-x}}\,dx$

18. $\int_0^\infty \frac{e^x}{1+e^x}\,dx$

19. $\int_0^\infty \cos \pi x\,dx$

20. $\int_0^\infty \sin\frac{x}{2}\,dx$

21. $\int_0^1 \frac{1}{x^2}\,dx$

22. $\int_0^1 \frac{1}{x}\,dx$

23. $\int_0^8 \frac{1}{\sqrt[3]{8-x}}\,dx$

24. $\int_0^e \ln x\,dx$

25. $\int_0^1 x\ln x\,dx$

26. $\int_0^{\pi/2} \sec\theta\,d\theta$

27. $\int_0^{\pi/2} \tan\theta\,d\theta$

28. $\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

29. $\int_2^4 \frac{1}{\sqrt{x^2-4}}\,dx$

30. $\int_0^2 \frac{1}{4-x^2}\,dx$

31. $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$

32. $\int_0^2 \frac{1}{(x-1)^{4/3}}\,dx$

33. Determine all values of p for which the following integral converges.

$$\int_1^\infty \frac{1}{x^p}\,dx$$

34. Determine all values of p for which the following integral converges.

$$\int_0^1 \frac{1}{x^p}\,dx$$

35. Use mathematical induction to verify that the following integral converges for any positive integer n.

$$\int_0^\infty x^ne^{-x}\,dx$$

36. Given continuous functions f and g such that $0 \le f(x) \le g(x)$ on the interval (a, ∞), prove the following.
(a) If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
(b) If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

In Exercises 37–46, use the results of Exercises 33–36 to determine whether the improper integral converges.

37. $\int_0^1 \frac{1}{x^3}\,dx$

38. $\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$

39. $\int_1^\infty \frac{1}{x^3}\,dx$

40. $\int_0^\infty x^4e^{-x}\,dx$

41. $\int_1^\infty \frac{1}{x^2+5}\,dx$

42. $\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$

43. $\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}}\,dx$

44. $\int_1^\infty \frac{1}{\sqrt{x(x+1)}}\,dx$

45. $\int_0^\infty e^{-x^2}\,dx$

46. $\int_2^\infty \frac{1}{\sqrt{x}\ln x}\,dx$

Let $f(t)$ be a function defined for all positive values of t. The Laplace Transform of $f(t)$ is defined by

$$F(s) = \int_0^\infty e^{-st}f(t)\,dt$$

if the improper integral exists. In Exercises 47–54, find the Laplace Transform of the function.

47. $f(t) = 1$

48. $f(t) = t$

49. $f(t) = t^2$

50. $f(t) = e^{at}$

51. $f(t) = \cos at$

52. $f(t) = \sin at$

53. $f(t) = \cosh at$

54. $f(t) = \sinh at$

55. Given the region bounded by the graphs of $y = 1/x^2$ and $y = 0$ $(x \ge 1)$, find the following.
(a) the area of the region
(b) the volume of the solid generated by revolving the region about the x-axis
(c) the volume of the solid generated by revolving the region about the y-axis

56. Given the region bounded by the graphs of $y = e^{-x}$ and $y = 0$ $(x \ge 0)$, find the following.
(a) the area of the region
(b) the volume of the solid generated by revolving the region about the y-axis