Math 21 Review for exam 2

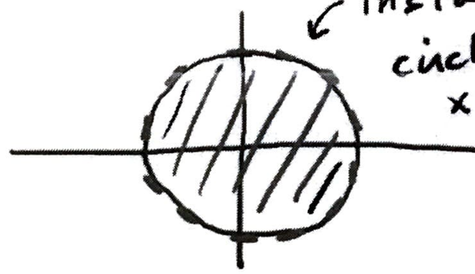
① Find the domain of $f(x,y)$.

A) $f(x,y) = \dfrac{1}{\sqrt{\underbrace{1-x^2-y^2}_{>0}}}$

$1-x^2-y^2 > 0$

$1 > x^2+y^2$

$x^2+y^2 < 1$



inside the circle $x^2+y^2=1$.

B) $f(x,y) = \ln(\underbrace{xy}_{})$

$xy > 0$

$xy>0 \Rightarrow x>0$ and $y>0$

or

$x<0$ and $y<0$

$x>0$ and $y>0$

$x<0$ and $y<0$

Q1 + Q3 without the x and y axes.

① Describe the level curves

A) $z = x^2 + y^2 + 5$ for $z=5$, $z=30$

$z=5$

$5 = x^2+y^2+5$

$0 = x^2+y^2 \Rightarrow$ point $(0,0)$

$z=30$

$30 = x^2+y^2+5$

$25 = x^2+y^2 \leftarrow$ circle centered at $(0,0)$ with radius 5

B) $z = 6-2x-3y$

$z=0$

$0 = 6-2x-3y$

$3y = -2x+6$

$y = -\frac{2}{3}x + 2 \leftarrow$ line, slope $-\frac{2}{3}$, y-int $(0,2)$

$z=6$

$6 = 6-2x-3y$

$3y = -2x$

$y = -\frac{2}{3}x \leftarrow$ line slope $\frac{-2}{3}$ thru $(0,0)$.

③ Find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}$

A) $f(x,y) = x^2+y^2-2x^2y+10x+10y$

$f_x = 2x - 4xy + 10$

$f_y = 2y - 2x^2 + 10$

$f_{xx} = 2 - 4y$

$f_{yy} = 2$

$f_{xy} = -4x$

B) $f(x,y) = e^{-(x^2+y^2)}$

$f_x = -2x\, e^{-(x^2+y^2)}$

$f_y = -2y\, e^{-(x^2+y^2)}$

$f_{xx} = -2x\,(-2x)\, e^{-(x^2+y^2)} + (-2)\, e^{-(x^2+y^2)}$

$= -4x^2 e^{-(x^2+y^2)} - 2e^{-(x^2+y^2)}$

$f_{yy} = -2y\,(-2y)\, e^{-(x^2+y^2)} + (-2)\, e^{-(x^2+y^2)}$

$f_{xy} = -2x\,(-2y)\, e^{-(x^2+y^2)} = -4xy\, e^{-(x^2+y^2)}$

④ Eqn of tangent plane to

$f(x,y) = 6x^2 - 2x + 3y$ at $(1,-1)$.
$\uparrow_{x_0} \uparrow_{y_0}$

$f_x = 12x - 2$ $\quad f_y = 3$

$f_x(1,-1) = 12 - 2 = 10$ $\quad f_y(1,-1) = 3$

tangent plane : $z - z_0 = 10(x-1) + 3(y+1)$
$\uparrow z_0 =$ $\quad \uparrow x_0$ $\quad \uparrow y_0$

$z_0 = f(x_0,y_0) = f(1,-1) =$
$6(1)^2 - 2(1) + 3(-1)$
$= 1$

$$\boxed{z - 1 = 10(x-1) + 3(y+1)}$$

⑤ $f(x,y) = x\cos y - y\cos x$ ← find the differential:
$f_x\,dx + f_y\,dy$

$f_x = \cos y + y\sin x$

$f_y = -x\sin y - \cos x$

$$f_x dx + f_y dy = \boxed{(\cos y + y\sin x)\,dx + (-x\sin y - \cos x)\,dy}$$

6 $f(x,y) = x^2y^2 - 3xy + 10x$

A) $\Delta z = f(1.05, 2.1) - f(1,2)$

$f(1.05, 2.1) = (1.05)^2(2.1)^2 - 3(1.05)(2.1) + 10(1.05)$

$= 8.747025$

$f(1,2) = (1)^2(2)^2 - 3(1)(2) + 10(1) = 8$

$\Delta z = 8.747025 - 8 = .747025$

b) Find $f_x\,dx + f_y\,dy$ at $(1,2)$, $dx = .05$, $dy = .1$

$f_x = 2xy^2 - 3y + 10$ $\quad$ $f_y = 2yx^2 - 3x$

$f_x(1,2) = 2(1)(2)^2 - 3(2) + 10$ $\quad$ $f_y(1,2) = 2(2)(1)^2 - 3(1)$

$= 12$ $\quad$ $= 1$

$f_x\,dx + f_y\,dy = (12)(.05) + (1)(.1)$

$= .6 + .1 = .7$

↑ close to .747025

⑦ Find $\frac{dz}{dt}$ $\left(z = f(x,y),\ x = x(t),\ y = y(t),\ \frac{dz}{dt} = f_x\, x'(t) + f_y\, y'(t) \right)$

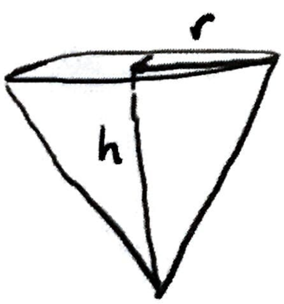
A) $z = \sqrt{x^2+y^2}$, $x = \sin t$, $y = e^t$

$$\frac{dz}{dt} = \frac{2x}{2\sqrt{x^2+y^2}}(\cos t) + \frac{2y}{2\sqrt{x^2+y^2}}(et)$$

B) $z = xy\cos x$, $x = t$, $y = t^2$

$$\frac{dz}{dt} = (-xy\sin x + y\cos x)(1) + (x\cos x)(2t)$$

⑧



$r = 3t$ $\qquad h = t^2$

Find $\frac{dV}{dt}$ when $t = 3$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = V_r \cdot r'(t) + V_h \cdot h'(t)$$

$$= \left(\frac{2}{3}\pi r h\right)(3) + \left(\frac{1}{3}\pi r^2\right)(2t)$$

when $t = 3$ $\leftarrow$ $r = 3(3) = 9$ $\quad h = (3)^2 = 9$

$$\frac{dV}{dt} = \left(\frac{2}{3}\pi(9)(9)\right)(3) + \left(\frac{1}{3}\pi(9)^2\right)(2\cdot 3)$$

$$= \frac{486\pi}{3} + \frac{162\pi}{1} = \boxed{324\pi} \ in^3/sec$$

9 $z = x^2 - y^2$, $x = s\cos t$ and $y = s\sin t$

A) $$\frac{dz}{ds} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$= \boxed{(2x)(\cos t) + (-2y)(\sin t)}$$

B) $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$
$$= \boxed{(2x)(-s\sin t) + (-2y)(s\cos t)}$$

10 Find min/max + saddle pts

A) $f(x,y) = \sqrt{x^2+y^2+1}$

$$f_x = \frac{2x}{2\sqrt{x^2+y^2+1}} = 0 \Rightarrow x = 0$$

$$f_y = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+1}} = 0 \Rightarrow y = 0$$

$$f_{xx} = \frac{\left(\sqrt{x^2+y^2+1}\right)(1) - \frac{x}{\sqrt{x^2+y^2+1}}\cdot x}{x^2+y^2+1}$$

$$= \frac{x^2+y^2+1 - x^2}{(x^2+y^2+1)\sqrt{x^2+y^2+1}} = \frac{y^2+1}{(x^2+y^2+1)\sqrt{x^2+y^2+1}}$$

$$f_{yy} = \frac{x^2+1}{(x^2+y^2+1)\sqrt{x^2+y^2+1}}$$

$$f_{xy} = -\tfrac{1}{2}\, x\,(x^2+y^2+1)^{-3/2}(2y)$$

$$= \frac{-xy}{(x^2+y^2+1)^{3/2}}$$

$$d = f_{xx} f_{xy} - (f_{xy})^2$$

$$d(0,0) = (1)(1) - 0 = 1 > 0 \text{ and } f_{xx} > 0$$

$$\Rightarrow \boxed{\text{min at } (0,0)}.$$

3) $f(x,y) = -x^2 - y^2 + 4x + 8y - 11$

$f_x = -2x + 4 = 0$ $\quad$ $f_y = -2y + 8 = 0$

$2x = 4$ $\quad$ $2y = 8$

$x = 2$ $\quad$ $y = 4$

$(2,4) \leftarrow$ cv

$f_{xx} = -2$ $\quad$ $f_{yy} = -2$ $\quad$ $f_{xy} = 0$

$d = (-2)(-2) - 0 = 4 > 0$ and $f_{xx} < 0 \Rightarrow$

$\boxed{(2,4) \text{ is a max}}$

C) $f(x,y) = -5x^2 + 4xy - y^2 + 16x + 10$

$f_x = -10x + 4y + 16 = 0$ $\quad f_y = 4x - 2y = 0$

$\uparrow 2x$ $\quad 2y = 4x$

$-10x + 8x + 16 = 0$ $\quad y = 2x$

$-2x + 16 = 0$

$16 = 2x$

$8 = x \Rightarrow y = 16$

$(8,16) \leftarrow$ cv

$f_{xx} = -10 \quad f_{yy} = -2 \quad f_{xy} = 4$

$d = (-10)(-2) - (4)^2 = 20 - 16 = 4 > 0$

and $f_{xx} < 0 \Rightarrow$ max at $(8,16)$

D) $f(x,y) = x^3 - 3xy + y^3$

$f_x = 3x^2 - 3y \qquad f_y = -3x + 3y^2$

$f_{xx} = 6x \qquad f_{yy} = 6y$

$f_{xy} = 0$

$f_x = 3x^2 - 3y = 0$ $\qquad\qquad$ $f_y = -3x + 3y^2 = 0$

$y = x^2$ $\qquad\qquad$ $y^2 = x$

$\uparrow$ y^2

$y = y^4$

$y - y^4 = 0$

$y(1 - y^3) = 0$

$y = 0$, $y = 1$

$x = 0$ $\quad$ $x = 1$

$(0,0)$ + $(1,1)$ $\leftarrow$ cv's

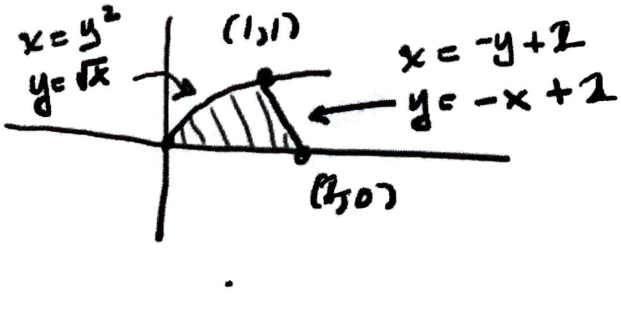
$d = (6x)(6y) - \cancel{9}\,(-3)^2 = 36xy - 9$

$d(0,0) = 0 \implies$ indeterminant

$d(1,1) = 36(1)(1) - 9 = 36 - 9$ and $f_{xx}(1,1) = 6 > 0$

$\implies$ $\boxed{(1,1) \text{ is a min.}}$

(11) Area bounded by $y = \sqrt{x}$, $y = -x + 2$, x-axis. (Use a double integral).



$$\int_0^1 \int_{y^2}^{-y+2} 1 \, dx \, dy$$

$$= \int_0^1 (-y+2) - (y^2)\, dy = \int_0^1 -y^2 - y + 2\, dy$$

$$= \left[\frac{-y^3}{3} - \frac{y^2}{2} + 2y \right]_0^1 = \frac{-1}{3} - \frac{1}{2} + 2 = \frac{-2}{6} - \frac{3}{6} + 2$$

$$= \boxed{\frac{7}{6}}$$

(12) A) $\int_0^3 \int_0^2 4 - y^2\, dy\, dx =$

$$\int_0^3 \left[4y - \frac{y^3}{3} \right]_0^2 dx = \int_0^3 \left[\overbrace{8 - \frac{8}{3}}^{\frac{16}{3}} \right] dx$$

$$= \left[\frac{16}{3} x \right]_0^3 = \frac{48}{3} = \boxed{16}$$

B) $\int_{-1}^0 \int_{-1}^1 x + y + 1\, dx\, dy =$

$$\int_{-1}^0 \left[\frac{x^2}{2} + yx + x \right]_{-1}^1 dy =$$

$$= \int_{-1}^0 \left(\frac{1}{2} + y + 1\right) - \left(\frac{1}{2} - y - 1\right) dy$$

$$= \int_{-1}^0 2y + 2\, dy = \left[y^2 + 2y \right]_{-1}^0 =$$

$$-(1-2) = \boxed{1}$$

(12) c) $\int_0^\pi \int_0^x x\sin y \, dy \, dx =$

$\int_0^\pi \left[-x\cos y\right]_0^x dx = \int_0^\pi \left[-x\cos x + x\right] dx =$

$\int x\cos x \, dx = x\sin x - \int \sin x \, dx = x\sin x + \cos x$

$u = x$ $\quad dv = \cos x \, dx$

$du = dx$ $\quad v = \sin x$

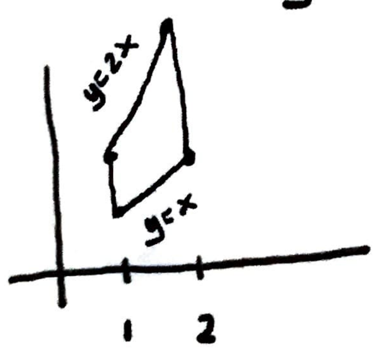
$-\int_0^\pi x\cos x \, dx + \left[\frac{x^2}{2}\right]_0^\pi$

$= -\left[x\sin x + \cos x\right]_0^\pi + \left[\frac{x^2}{2}\right]_0^\pi$

$= -\left[\pi(0) - 1 - 1\right] + \frac{\pi^2}{2}$

$= \boxed{2 + \frac{\pi^2}{2}}$

(13) $f(x,y) = \frac{x}{y}$; integrate over region:



$\int_1^2 \int_x^{2x} \frac{x}{y} \, dy \, dx =$

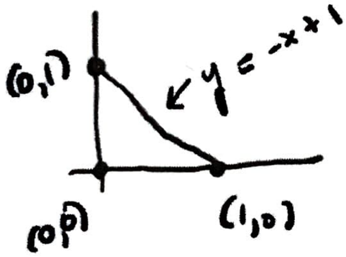
$\int_1^2 \left[x\ln y\right]_x^{2x} dx =$

$$= \int_1^2 \left[x \overbrace{\ln(2x)}^{\ln 2 + \ln x} - x\ln(x)\right] dx = \int_1^2 x(\ln 2) + \cancel{x \ln x} - \cancel{x \ln x}\, dx$$

$$= \left[(\ln 2)\frac{x^2}{2}\right]_1^2 = 2(\ln 2) - \frac{1}{2}(\ln 2) = \boxed{\frac{3}{2}(\ln 2)}$$

(14)

$$\int_0^1 \int_0^{-x+1} x^2 + y^2 \; dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{-x+1} dx$$

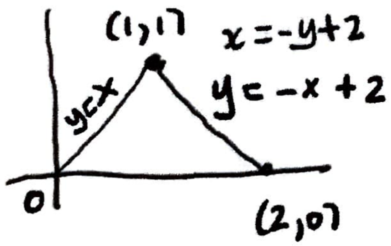


$$= \int_0^1 \overbrace{x^2(-x+1)}^{-x^3+x^2} + \frac{(-x+1)^3}{3} \; dx$$

$$= \left[\frac{-x^4}{4} + \frac{x^3}{3} - \frac{(-x+1)^4}{12}\right]_0^1$$

$$= \left[\underbrace{-\frac{1}{4} + \frac{1}{3}}_{\frac{1}{12}}\right] - \left[-\frac{1}{12}\right] = \frac{2}{12} = \boxed{\frac{1}{6}}$$

(15) $f(x,y) = x^2 + y^2$; Vol below $f(x,y)$ + above region bounded by $y = x$, $x = 0$, $x + y = 2$.



$$\int_0^1 \int_y^{-y+2} x^2 + y^2 \; dx\, dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_y^{-y+2} dy =$$

$$\int_0^1 \left[\frac{(-y+2)^3}{3} + \overbrace{y^2(-y+2)}^{-y^3+2y^2}\right] - \left[\frac{y^3}{3} + y^3\right] dy$$

$$\int_0^1 \frac{(-y+2)^3}{3} - y^3 + 2y^2 - \frac{y^3}{3} - y^3 \, dy$$

$$\int_0^1 \frac{(-y+2)^3}{3} + 2y^2 - \frac{7y^3}{3} \, dy$$

$$\left[\frac{-(-y+2)^4}{12} + \frac{2y^3}{3} - \frac{7y^4}{12}\right]_0^1$$

$$= \left[-\frac{1}{12} + \frac{2}{3} - \frac{7}{12}\right] - \left[-\frac{16}{12}\right]$$

$$= \frac{-1}{12} + \frac{8}{12} - \frac{7}{12} + \frac{16}{12} = \frac{16}{12} = \boxed{\frac{4}{3}}$$

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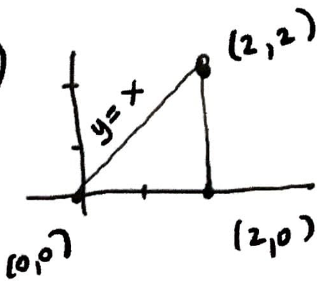

$f(x,y) = 2x$

$$\int_0^2 \int_0^x 2x \, dy \, dx$$

$$= \int_0^2 [2xy]_0^x \, dx = \int_0^2 [2x^2] \, dx$$

$$= \left[\frac{2x^3}{3}\right]_0^2 = \frac{2(8)}{3} = \boxed{\frac{16}{3}}$$